Introduction to Complexity Theory – CS-28100
Homework 1 – March 29, 2006
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HOMEWORK. Please **print your name on each sheet.** Please try to make your solutions readable.

This homework is due on **Wednesday, April 5** at the **beginning of the class.**

1.1 `Kruskal's Algorithm`

Sort the edges of G according to weight
Let $T = \emptyset$.
while $(|T| < n - 1)$ **do**
    Pick the edge of lowest weight in G and see if it creates a cycle
    If it does, delete it
    If it doesn't, add it to T and delete it
end (**while**)
return (T)

Prove that Kruskal's Algorithm returns a minimum spanning tree. (Hint: By construction. Find largest k such that the first k edges are part of a minimum spanning tree.)

1.2 Consider the following two problems:

- `Minimum Spanning Tree`
  Given a weighted graph G, find a spanning tree of minimum total weight.
- `Minimum Perfect Matching`
  Given a weighted graph G, we define a perfect matching to be a subset of the vertices such that: (i) no two of the selected edges share a vertex, and (ii) every vertex of G is incident to one of the selected edges. Find a perfect matching of minimum total weight.

Show that we can formulate Minimum Spanning Tree (for any graph) and Minimum Perfect Matching (for bipartite graphs) as linear programming questions. (Hint: First prove that we can formulate them as integer linear programming questions. For this create a variable for each edge and express the total weight as a function of these variables. For constraints defining the problems state them mathematically, as you find them below. For both problems, there are two types of constraints. One is on the number of total edges included in the set and the other is on the number of total edges incident to a given vertex. Aditionally, there is a third type of constraint for spanning tree problem: You must avoid cycles, so for each cycle the number of edges of the cycle included in the tree must be bounded. Using these set of constraints you can also assume that the polytope formed by these constraints has integral vertices, which leads to the solution of the non-integral relaxation of the problem.)

1.3 (This problem is optional, and can be done for extra credit.) Prove that if an integer linear program has a solution, then it has a solution of polynomial size (bit-length) in the input size. (**Hint:** Consider the Hermite normal form of the matrix which specifies the constraints.)

1.4 Show that every Boolean formula is equivalent to one in conjunctive normal form. (Homework 1.3 from the book of Homer and Selman)

1.5 The set $\{\langle x, y\rangle \mid x, y \in \mathbb{N}\}$ is countable by the enumeration
$\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,0\rangle \ldots$
Find a formula for the i^{th} enumerated element, i.e., $\langle x_i, y_i\rangle$.
Also find a formula for the inverse mapping, *i.e.* find the stage at which $\langle x, y\rangle$ is enumerated.