1a

$$M_0 = (\lambda g. g(\bar{f}\, z))((\lambda y. \lambda z. \lambda x. z(\lambda u. u + x))\, \bar{x}\, (\lambda f. f\, \bar{y}))$$

$$M_1 = \lambda y. \bar{x}(\lambda z. \lambda x. y(\lambda z. z\, x))\bar{z}$$

1b $\{g/f\} M_0 =$

$$(\lambda g'. g'(g\, z))((\lambda y. \lambda z. \lambda y. z(\lambda u. u + x))\, x\, (\lambda f. f\, y))$$

α-conversion before subst.

$\{f/g\} M_0 = M_0$ – no change, $g \notin FV(M_0)$

$FV(M_0) = \{f, x, y\}$

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2. [FV = FN = free variables]

Proof: $\forall \Gamma. \forall e. \Gamma \vdash e \text{ ok} \iff FV(e) \subseteq \Gamma$

($\Rightarrow$) $\forall \Gamma. \forall e. \Gamma \vdash e \text{ ok} \Rightarrow FV(e) \subseteq \Gamma$.

Assume, for some Γ, e, that $\Gamma \vdash e$ ok. We prove $FV(e) \subseteq \Gamma$ by ind. on derivation of $\Gamma \vdash e$ ok.

Base Case 1: $\Gamma \vdash e$ ok by rule S1 (the Var rule).

Then $e = x$ for some variable x, and $x \in \Gamma$.

Then $FV(e) = \{x\}$ (defn of FV), and $\{x\} \subseteq \Gamma$. □

Base Case 2: $\Gamma \vdash e$ ok by rule S2 (the Num rule).

Then $e = \underline{n}$ for some $n \in \mathbb{N}$. Thus $FV(e) = FV(\underline{n}) = \emptyset \subseteq \Gamma$. □

Ind Case 1: $\Gamma \vdash e$ ok by rule S3 (Plus rule).

Then $e = \text{plus}(e_1, e_2)$. By inversion of the rule, we have (1) $\Gamma \vdash e_1$ ok and (2) $\Gamma \vdash e_2$ ok.

The Ind. Hypothesis is

$$\text{(IH1)}\quad \Gamma \vdash e_1\ \text{ok} \Rightarrow FV(e_1) \subseteq \Gamma \qquad \text{(same } \Gamma\text{)}$$

$$\&\ \text{(IH2)}\quad \Gamma \vdash e_2\ \text{ok} \Rightarrow FV(e_2) \subseteq \Gamma \qquad \text{"}$$

By (1) and (IH1), we have (3) $FV(e_1) \subseteq \Gamma$ and (4) $FV(e_2) \subseteq \Gamma$. Thus

$$FV(e) = FV(e_1) \cup FV(e_2) \subseteq \Gamma \quad \square$$

The case for the Times rule is similar. (Ind. Case 2)

<u>Ind Case 3</u>: $\Gamma \vdash e$ ok by rule S5 (Let rule).

Then $e = \text{let}(e_1, x.e_2)$. By inversion of the Let rule we have

(1) $\Gamma \vdash e_1$ ok

(2) $\Gamma \cup \{x\} \vdash e_2$ ok

The ind. hyp. is

$$\text{(IH1)}\quad \Gamma \vdash e_1\ \text{ok} \Rightarrow FV(e_1) \subseteq \Gamma$$

$$\&\ \text{(IH2)}\quad \Gamma \cup \{x\} \vdash e_2\ \text{ok} \Rightarrow FV(e_2) \subseteq \Gamma \cup \{x\}$$

Then by (1) and (IH1), and (2) and (IH2) resp.

(3) $FV(e_1) \subseteq \Gamma$

(4) $FV(e_2) \subseteq \Gamma \cup \{x\}$ $\quad (\Rightarrow FV(e_2) \setminus \{x\} \subseteq \Gamma)$

which implies

$$\begin{aligned}(5)\quad FV(e) &= FV(\text{let}(e_1, x.e_2)) \\ &= FV(e_1) \cup (FV(e_2) \setminus \{x\}) \\ &\subseteq \Gamma\end{aligned}$$

$\boxtimes$

2. ($\Leftarrow$) $\forall e. \forall \Gamma. FV(e) \subseteq \Gamma \Rightarrow \Gamma \vdash e$ ok

Assume $FV(e) \subseteq \Gamma$. Prove $\Gamma \vdash e$ ok by ind. on structure of e.

Base Case 1: $e = x$, a variable.

Then $FV(e) = \{x\}$, so $\{x\} \subseteq \Gamma$ means $x \in \Gamma$. Thus by rule S1, $\Gamma \vdash e$ ok. $\square$

Base Case 2: $e = n$, a number. Then $\Gamma \vdash e$ ok by rule Num.

Ind. Case 1: $e = plus(e_1, e_2)$. Then $FV(e) = FV(e_1) \cup FV(e_2)$.

So $FV(e) \subseteq \Gamma \Rightarrow$

(1) $FV(e_1) \subseteq \Gamma$

(2) $FV(e_2) \subseteq \Gamma$

The Ind. Hyp. is

(IH1) $\forall \Gamma'. FV(e_1) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_1$ ok

& (IH2) $\forall \Gamma'. FV(e_2) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_2$ ok

Then

(3) $\Gamma \vdash e_1$ ok by (1) and (IH1)

(4) $\Gamma \vdash e_2$ ok by (2) and (IH2)

Then by (3) and (4) and rule Plus (S3), we have $\Gamma \vdash e$ ok.

The Times case is similar. (Ind Case 2)

Ind Case 3: $e = let(e_1, x.e_2)$.

Then $FV(e) = FV(e_1) \cup (FV(e_2) \setminus \{x\})$. $FV(e) \subseteq \Gamma$ implies

(1) $FV(e_1) \subseteq \Gamma$

(2) $FV(e_2) \setminus \{x\} \subseteq \Gamma$ ($\Rightarrow FV(e_2) \subseteq \Gamma \cup \{x\}$)

The Ind Hyp is

(IH1) $\forall \Gamma'.\ FV(e_1) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_1 \text{ ok}$

& (IH2) $\forall \Gamma'.\ FV(e_2) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_2 \text{ ok}$

Then

(3) $\Gamma \vdash e_1 \text{ ok}$ by (1) and (IH1)

(4) $\Gamma \cup \{x\} \vdash e_2 \text{ ok}$ by (2) and (IH2)

Then (3) and (4) and the Let rule give $\Gamma \vdash e \text{ ok}$. $\boxtimes$

3. $e \mapsto e_1$ and $e \mapsto e_2 \Rightarrow e_1 = e_2$.

Proof by induction on the derivation of $e \mapsto e_1$ (i.e. by rule induction on the rules for $\mapsto$).

Base Case 1. $e \mapsto e_1$ by E_1, the plus instruction.

Then $e = \text{plus}(m, n)$ and $e_1 = p$ where $p = m+n$.

Then the only rule by whice $e \mapsto e_2$ is E_1, so e_2 must be p as well. Hence $e_1 = p = e_2$. □

Similarly for E2.

Base Case 2: $e \mapsto e_1$ by E_3, the let instruction.

Then $e = \text{let}(n, x.e')$ and $e_1 = \{n/x\}e'$.

The only rule that matches e is E_3, so $e \mapsto e_2$ by the same rule, so $e_2 = \{n/x\}e' = e_1$. □

Ind. Case 1. $e \mapsto e_1$ by E_4, the left plus search rule.

Then $e = \text{plus}(e_3, e_4)$ and $e_1 = \text{plus}(e_3', e_4)$ where $e_3 \mapsto e_3'$. The induction hypothesis is:

$$(IH)\ \forall f_1, f_2 .\ e_3 \mapsto f_1 \ \&\ e_3 \mapsto f_2 \Rightarrow f_1 = f_2$$

The only rule by which $e \mapsto e_2$ can be deduced is E_4, so $\exists e_5 .\ e_2 = \text{plus}(e_5, e_4)$, and $e_3 \mapsto e_5$.

But by the (IH), taking $f_1 = e_3'$ and $f_2 = e_5$, we have $e_3' = e_5$. But then $e_2 = \text{plus}(e_5, e_4) = \text{plus}(e_3', e_4) = e_1$. □

All the other search rule cases follow the same pattern, with minor, and obvious changes.