Homework 6

Exercise 0.1 *Show that the CFL's are closed under the following operations:*

1. *init, defined as* $init(L) = \{w : \text{for some } x, wx \text{ is in } L\}$. *Hint: Start with a CNF grammar for the language* L.
2. *The operation* L/a, *defined as* $L/a = \{w : wa \in L\}$. *For example, if* $L = \{a, aab, baa\}$, *then* $L/a = \{\varepsilon, ba\}$. *Hint: Again, start with a CNF grammar for* L.
3. *cycle, defined as* $cycle(L) = \{w : \text{we can write } w \text{ as } w = xy, \text{ such that } yx \text{ is in } L\}$. *For example, if* $L = \{01, 011\}$, *then* $cycle(L) = \{01, 10, 011, 110, 101\}$. *Hint: Try a PDA-based construction.*

Exercise 0.2 *Give the formal proof of the following theorem (Thm 7.25 of the textbook): that the CFL's are closed under reversal.*

Exercise 0.3 *The shuffle of two strings* w *and* x *is the set of all strings that one can get by interleaving the positions of* w *and* x *in any way. More precisely,* $shuffle(w, x)$ *is the set of strings* z *such that*

1. *Each position of* z *can be assigned to* w *or* x, *but not both.*
2. *The positions of* z *assigned to* w *form* w *when read from left to right.*
3. *The positions of* z *assigned to* x *form* x *when read from left to right.*

For example, if $w = 01$ *and* $x = 110$, *then*

$$shuffle(01, 110) = \{01110, 01101, 10110, 10101, 11010, 11001\}.$$

To illustrate the necessary reasoning, the fourth string, 10101, *is justified by assigning the second and fifth positions to* 01 *and positions one, three, and four to* 110. *The first string,* 01110, *has three justifications. Assign the first position and either the second, third, or fourth to* 01, *and the other three to* 110.

We can also define the shuffle of languages, $shuffle(L_1, L_2)$, *to be the union over all pairs of strings,* w *from* L_1 *and* x *from* L_2, *of* $shuffle(w, x)$.

1. *What is* $shuffle(00, 111)$?
2. *What is* $shuffle(L_1, L_2)$ *if* $L_1 = L(\mathbf{0}^*)$ *and* $L_2 = \{0^n 1^n : n \geq 0\}$?
3. *Show that if* L_1 *and* L_2 *are both regular languages, then so is* $shuffle(L_1, L_2)$. *Hint: Start with DFA's for* L_1 *and* L_2.

4. *Show that if L is a CFL and R is a regular language, then shuffle(L, R) is a CFL. Hint: start with a PDA for L and a DFA for R.*

5. *Give a counterexample to show that if L_1 and L_2 are both CFL's, then shuffle(L_1, L_2) need not be a CFL.*

Exercise 0.4 *Design Turing machines for the following languages:*

1. *The set of strings with an equal number of 0's and 1's.*

2. $\{a^n b^n c^n : n \geq 1\}$

3. $\{ww^R : w$ *is any string of* 0*'s and* 1*'s*$\}$.

* Exercises above are from *Introduction to Automata Theory, Languages, and Computation, 3rd Edition*: Exercises 7.3.1, 7.3.6, 7.3.4, 8.2.2