CMSC 28100-1 Spring 2015
Homework 6

1. Let the complexity class $\mathbf{P}$ be the class of all languages decidable in polynomial time. In other words, $\mathbf{P} = \cup_{k \geqslant 1}\text{TIME}(n^k)$. Prove that $\mathbf{P} \neq \text{TIME}(n^3)$.

2. The number of *head reversals* of a single-tape Turing machine M on input x is the number of times the tape head changes direction – that is, it was moving left and it moves right, or vice versa. (Not moving does not count as reversing direction, but e.g. moving right, not moving, then moving left does.)

Let REVERSAL$(r(n))$ denote the class of languages decidable by Turing machines that on inputs of length n use at most $r(n)$ head reversals.

Prove the following head reversal hierarchy: If $\lim_{n\to\infty} \frac{r(n)^2}{R(n)} = 0$ then there is a language in REVERSAL$(R(n))$ that is not in REVERSAL$(r(n))$. [[Note that previous version had a typo: REVERSAL$(r(n)^2)$ is now changed to REVERSAL$(r(n))$]]

3. (Based on Exercise 9.13 of Sipser's "Introduction to Theory of Computation") Consider the function $pad : \Sigma^* \times \mathbb{N} \to \Sigma^*\#^*$ that is defined as follows. Let $pad(s, l) = s\#^j$ where $j = \max\{0, l - m\}$ and m is the length of s. Thus $pad(s, l)$ simply adds enough copies of the new symbol # to the end of s so that the length of the result is at least l. For any language A and function $f : \mathbb{N} \to \mathbb{N}$, define the language $pad(A, f(m))$ as

$$pad(A, f(m)) = \{pad(s, f(m)) \mid \text{where } s \in A \text{ and } m \text{ is the length of } s\}.$$

(a) Prove that if $A \in \text{TIME}(n^6)$ then $pad(A, n^2) \in \text{TIME}(n^3)$.

(b) Let Fac be the language

$$\{(x, y) \mid x \text{ and } y \text{ are written in binary, and } x \text{ has a nontrivial factor} \leqslant y\}$$

A nontrivial factor of x is a factor that is neither 1 nor x. Show that Fac can be decided in time $O(x(\log x)^2) = O(2^n n^2)$ where n is the length of x.

(c) Let $UnaryFac$ be the language

$$\{(x, y) \mid x \text{ and } y \text{ are written in } \textit{unary}\text{, and } x \text{ has a nontrivial factor} \leqslant y\}$$

Show that $UnaryFac \in \text{TIME}(n(\log n)^2)$, where n is the length of x. (The *unary* representation of an integer x is simply a string of x many 1's.)

(d) Show that $pad(Fac, 2^n) \in \text{TIME}(n(\log n)^2)$.