For instance, the sum of the indegrees of the vertices of a digraph is equal to the total number of arcs (Exercise 1.5.2). Applying the Principle of Directional Duality, we immediately deduce that the sum of the outdegrees is also equal to the number of arcs.

Apart from the practical aspect mentioned earlier, assigning suitable orientations to the edges of a graph is a convenient way of exploring properties of the graph, as we shall see in Chapter 6.

Exercises

1.5.1 How many orientations are there of a labelled graph G?

$\star$**1.5.2** Let D be a digraph.

a) Show that $\sum_{v \in V} d^-(v) = m$.
b) Using the Principle of Directional Duality, deduce that $\sum_{v \in V} d^+(v) = m$.

1.5.3 Two digraphs D and D' are *isomorphic*, written $D \cong D'$, if there are bijections $\theta : V(D) \to V(D')$ and $\phi : A(D) \to A(D')$ such that $\psi_D(a) = (u, v)$ if and only if $\psi_{D'}(\phi(a)) = (\theta(u), \theta(v))$. Such a pair of mappings is called an *isomorphism* between D and D'.

a) Show that the four tournaments in Figure 1.25 are pairwise nonisomorphic, and that these are the only ones on four vertices, up to isomorphism.
b) How many tournaments are there on five vertices, up to isomorphism?

1.5.4

a) Define the notions of vertex-transitivity and arc-transitivity for digraphs.
b) Show that:
   i) every vertex-transitive digraph is diregular,
   ii) the Koh–Tindell digraph (Figure 1.26a) is vertex-transitive but not arc-transitive.

1.5.5 A digraph is *self-converse* if it is isomorphic to its converse. Show that both digraphs in Figure 1.26 are self-converse.

$\star$**1.5.7** Totally Unimodular Matrix
A matrix is *totally unimodular* if each of its square submatrices has determinant equal to 0, +1, or −1. Let $\mathbf{M}$ be the incidence matrix of a digraph.

a) Show that $\mathbf{M}$ is totally unimodular. (H. Poincaré)
b) Deduce that the matrix equation $\mathbf{Mx} = \mathbf{b}$ has a solution in integers provided that it is consistent and the vector $\mathbf{b}$ is integral.

1.5.8 Balanced Digraph
A digraph D is *balanced* if $|d^+(v) - d^-(v)| \leq 1$, for all $v \in V$. Show that every graph has a balanced orientation.

1.5.9 Describe how the two digraphs in Figure 1.26 can be constructed from the Fano plane.

1.5.10 Paley Tournament
Let q be a prime power, $q \equiv 3 \pmod 4$. The *Paley tournament* PT_q is the tournament whose vertex set is the set of elements of the field $GF(q)$, vertex i dominating vertex j if and only if $j - i$ is a nonzero square in $GF(q)$.

a) Draw PT_3, PT_7, and PT_{11}.
b) Show that these three digraphs are self-converse.

1.5.11 Stockmeyer Tournament
For a nonzero integer k, let $\mathrm{pow}(k)$ denote the greatest integer p such that 2^p divides k, and set $\mathrm{odd}(k) := k/2^p$. (For example, $\mathrm{pow}(12) = 2$ and $\mathrm{odd}(12) = 3$, whereas $\mathrm{pow}(-1) = 0$ and $\mathrm{odd}(-1) = -1$.) The *Stockmeyer tournament* ST_n, where $n \geq 1$, is the tournament whose vertex set is $\{1, 2, 3, \ldots, 2^n\}$ in which vertex i dominates vertex j if $\mathrm{odd}(j - i) \equiv 1 \pmod 4$.

a) Draw ST_2 and ST_3.
b) Show that ST_n is both self-converse and asymmetric (that is, has no nontrivial automorphisms). (P.K. Stockmeyer)

1.5.12 Arc-transitive Graph

to saying that if n integers sum to $n+1$ or more, one of them is at least $\lceil (n+1)/n \rceil = 2$.

Exercise 1.1.6a is a simple example of a statement that can be proved by applying this principle. As a second application, we establish a sufficient condition for the existence of a quadrilateral in a graph, due to Reiman (1958).

Theorem 2.2 *Any simple graph G with $\sum_{v\in V}\binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.*

Proof Denote by p_2 the number of paths of length two in G, and by $p_2(v)$ the number of such paths whose central vertex is v. Clearly, $p_2(v) = \binom{d(v)}{2}$. As each path of length two has a unique central vertex, $p_2 = \sum_{v\in V} p_2(v) = \sum_{v\in V}\binom{d(v)}{2}$. On the other hand, each such path also has a unique pair of ends. Therefore the set of all paths of length two can be partitioned into $\binom{n}{2}$ subsets according to their ends. The hypothesis $\sum_{v\in V}\binom{d(v)}{2} > \binom{n}{2}$ now implies, by virtue of the Pigeonhole Principle, that one of these subsets contains two or more paths; that is, there exist two paths of length two with the same pair of ends. The union of these paths is a quadrilateral. □

Exercises

⋆**2.1.1** Show that the maximal connected subgraphs of a graph are its components.

⋆**2.1.2**

a) Show that every nontrivial acyclic graph has at least two vertices of degree less than two.
b) Deduce that every nontrivial connected acyclic graph has at least two vertices of degree one. When does equality hold?

length greater than $k+1$.

2.1.6 Show that every simple graph has a vertex x and a family of $\lfloor \frac{1}{2}d(x)\rfloor$ cycles any two of which meet only in the vertex x.

2.1.7

a) Show that the Petersen graph has girth five and circumference nine.
b) How many cycles are there of length k in this graph, for $5 \le k \le 9$?

2.1.8

a) Show that a k-regular graph of girth four has at least $2k$ vertices.
b) For $k \ge 2$, determine all k-regular graphs of girth four on exactly $2k$ vertices.

2.1.9

a) Show that a k-regular graph of girth five has at least k^2+1 vertices.
b) Determine all k-regular graphs of girth five on exactly k^2+1 vertices, $k = 2, 3$.

2.1.10 Show that the incidence graph of a finite projective plane has girth six.

⋆**2.1.11** A *topological sort* of a digraph D is an linear ordering of its vertices such that, for every arc a of D, the tail of a precedes its head in the ordering.

a) Show that every acyclic digraph has at least one source and at least one sink.
b) Deduce that a digraph admits a topological sort if and only if it is acyclic.

2.1.12 Show that every strict acyclic digraph contains an arc whose reversal results in an acyclic digraph.

2.1.13 Let D be a strict digraph. Setting $k := \max\{\delta^-, \delta^+\}$, show that:

a) D contains a directed path of length at least k,
b) if $k > 0$, then D contains a directed cycle of length at least $k+1$.

FIND: *a minimum-weight Hamilton cycle of G.*

Note that it suffices to consider the TSP for complete graphs because nonadjacent vertices can be joined by edges whose weights are prohibitively high. We discuss this problem, and others of a similar flavour, in Chapters 6 and 8, as well as in later chapters.

Exercises

2.2.1 Let G be a graph on n vertices and m edges and c components.

a) How many spanning subgraphs has G?
b) How many edges need to be added to G to obtain a connected spanning supergraph?

⋆**2.2.2**

a) Deduce from Theorem 2.4 that every loopless graph G contains a spanning bipartite subgraph F with $e(F) \geq \frac{1}{2}e(G)$.
b) Describe an algorithm for finding such a subgraph by first arranging the vertices in a linear order and then assigning them, one by one, to either X or Y, using a simple rule.

2.2.3 Determine the number of 1-factors in each of the following graphs: (a) the Petersen graph, (b) the pentagonal prism, (c) K_{2n}, (d) $K_{n,n}$.

2.2.4 Give a proof of Theorem 2.3 by means of a longest path argument.

(D. KÖNIG AND P. VERESS)

2.2.5

a) Show that every Hamilton cycle of the k-prism uses either exactly two consecutive edges linking the two k-cycles or else all of them.
b) How many Hamilton cycles are there in the pentagonal prism?

2.2.6 Show that there is a Hamilton path between two vertices in the Petersen graph if and only if these vertices are nonadjacent.

2.2.7
Which grids have Hamilton paths, and which have Hamilton cycles?

path.

2.2.9 Let G be a graph on n vertices and m edges.

a) How many induced subgraphs has G?
b) How many edge-induced subgraphs has G?

2.2.10 Show that every shortest cycle in a simple graph is an induced subgraph.

⋆**2.2.11** Show that if G is simple and connected, but not complete, then G contains an induced path of length two.

⋆**2.2.12** Let P and Q be distinct paths in a graph G with the same initial and terminal vertices. Show that $P \cup Q$ contains a cycle by considering the subgraph $G[E(P) \triangle E(Q)]$ and appealing to Theorem 2.1.

2.2.13

a) Show that any two longest paths in a connected graph have a vertex in common.
b) Deduce that if P is a longest path in a connected graph G, then no path in $G - V(P)$ is as long as P.

2.2.14 Give a constructive proof of Theorem 2.5.

2.2.15

a) Show that an induced subgraph of a line graph is itself a line graph.
b) Deduce that no line graph can contain either of the graphs in Figure 1.19 as an induced subgraph.
c) Show that these two graphs are minimal with respect to the above property. Can you find other such graphs? (There are nine in all.)

2.2.16

a) Show that an induced subgraph of an interval graph is itself an interval graph.
b) Deduce that no interval graph can contain the graph in Figure 1.20 as an induced subgraph.
c) Show that this graph is minimal with respect to the above property.

2.2.17 Let G be a bipartite graph of maximum degree k.

a) Show that there is a k-regular bipartite graph H which contains G as an the case of loopless digraphs, we refer to $|\partial^+(X)|$ and $|\partial^-(X)|$ as the *outdegree* and *indegree* of X, and denote these quantities by $d^+(X)$ and $d^-(X)$, respectively.

A digraph D is called *strongly connected* or *strong* if $\partial^+(X) \neq \emptyset$ for every nonempty proper subset X of V (and thus $\partial^-(X) \neq \emptyset$ for every nonempty proper subset X of V, too).

Exercises

⋆2.5.1

a) Prove Theorem 2.9.
b) Prove Proposition 2.13.
c) Deduce Theorem 2.14 from Proposition 2.11.

⋆2.5.2 Let D be a digraph, and let X be a subset of V.

a) Show that $|\partial^+(X)| = \sum_{v \in X} d^+(v) - a(X)$.
b) Suppose that D is even. Using the Principle of Directional Duality, deduce that $|\partial^+(X)| = |\partial^-(X)|$.
c) Deduce from (b) that every connected even digraph is strongly connected.

2.5.3 Let G be a graph, and let X and Y be subsets of V. Show that $\partial(X \cup Y) \triangle \partial(X \cap Y) = \partial(X \triangle Y)$.

⋆2.5.4 Let G be a loopless graph, and let X and Y be subsets of V.

a) Show that:

$$d(X) + d(Y) = d(X \cup Y) + d(X \cap Y) + 2e(X \setminus Y, Y \setminus X)$$

b) Deduce the following *submodular inequality* for degrees of sets of vertices.

$$d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y)$$

c) State and prove a directed analogue of this submodular inequality.

⋆2.5.5 An *odd graph* is one in which each vertex is of odd degree. Show that a graph G is odd if and only if $|\partial(X)| \equiv |X| \pmod 2$ for every subset X of V.

⋆2.5.6 Show that each arc of a strong digraph is contained in a directed cycle.

2.5.7 Directed Bond

ii) a digraph is strong if and only if no bond is a directed bond.

⋆2.5.8 Feedback Arc Set
A *feedback arc set* of a digraph D is a set S of arcs such that $D \setminus S$ is acyclic. Let S be a minimal feedback arc set of a digraph D. Show that there is a linear ordering of the vertices of D such that the arcs of S are precisely those arcs whose heads precede their tails in the ordering.

2.5.9 Let (D, w) be a weighted oriented graph. For $v \in V$, set $w^+(v) := \sum\{w(a) : a \in \partial^+(v)\}$. Suppose that $w^+(v) \geq 1$ for all $v \in V \setminus \{y\}$, where $y \in V$. Show that D contains a directed path of weight at least one, by proceeding as follows.

a) Consider an arc $(x, y) \in \partial^-(y)$ of maximum weight. Contract this arc to a vertex y', delete all arcs with tail y', and replace each pair $\{a, a'\}$ of multiple arcs (with head y') by a single arc of weight $w(a) + w(a')$, all other arcs keeping their original weights. Denote the resulting weighted digraph by (D', w'). Show that if D' contains a directed path of weight at least one, then so does D.
b) Deduce, by induction on V, that D contains a directed path of weight at least one. (B. Bollobás and A.D. Scott)

2.6 Even Subgraphs

By an *even subgraph* of a graph G we understand a *spanning* even subgraph of G, or frequently just the edge set of such a subgraph. Observe that the first two subgraphs in Figure 2.4 are both even, as is their symmetric difference. Indeed, it is an easy consequence of Proposition 2.13 that the symmetric difference of even subgraphs is always even.

Corollary 2.16 *The symmetric difference of two even subgraphs is an even subgraph.*

Proof Let F_1 and F_2 be even subgraphs of a graph G, and let X be a subset of V. By Proposition 2.13,

$$\partial_{F_1 \triangle F_2}(X) = \partial_{F_1}(X) \triangle \partial_{F_2}(X)$$

By Theorem 2.10, $\partial_{F_1}(X)$ and $\partial_{F_2}(X)$ are both of even cardinality, so their symmetric difference is too. Appealing again to Theorem 2.10, we deduce that $F_1 \triangle F_2$ is even. □