**Fig. 3.5.** Choosing a cut edge in Fleury's Algorithm

he validity of Fleury's Algorithm provides the following characterization of ian graphs.

**orem 3.5** *A connected graph is eulerian if and only if it is even.* □

et now $x$ and $y$ be two distinct vertices of a graph $G$. Suppose that we wish d an Euler $xy$-trail of $G$, if one exists. We may do so by adding a new edge $e$ ng $x$ and $y$. The graph $G$ has an Euler trail connecting $x$ and $y$ if and only if has an Euler tour (Exercise 3.3.3). Thus Fleury's Algorithm may be adapted to find an Euler $xy$-trail in $G$, if one exists.

Ve remark that Fleury's Algorithm is an efficient algorithm, in a sense to be precise in Chapter 8. When an edge is considered for inclusion in the current $W$, it must be examined to determine whether or not it is a cut edge of the ining subgraph $F$. If it is not, it is appended to $W$ right away. On the other , if it is found to be a cut edge of $F$, it remains a cut edge of $F$ until it is ually selected for inclusion in $W$; therefore, each edge needs to be examined once. In Chapter 7, we present an efficient algorithm for determining whether t an edge is a cut edge of a graph.

comprehensive treatment of eulerian graphs and related topics can be found eischner (1990, 1991).

## rcises

Which of the pictures in Figure 3.6 can be drawn without lifting one's pen the paper and without tracing a line more than once?

be the graph obtained from $G$ by the addition of a new edge $e$ joining $x$ and $y$.

a) Show that $G$ has an Euler trail connecting $x$ and $y$ if and only if $G + e$ has an Euler tour.
b) Deduce that $G$ has an Euler trail connecting $x$ and $y$ if and only if $d(x)$ and $d(y)$ are odd and $d(v)$ is even for all $v \in V \setminus \{x, y\}$.

**3.3.4** Let $G$ be a connected graph, and let $X$ be the set of vertices of $G$ of odd degree. Suppose that $|X| = 2k$, where $k \geq 1$.

a) Show that there are $k$ edge-disjoint trails $Q_1, Q_2, \ldots, Q_k$ in $G$ such that $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$.
b) Deduce that $G$ contains $k$ edge-disjoint paths connecting the vertices of $X$ in pairs.

**3.3.5** Let $G$ be a nontrivial eulerian graph, and let $v \in V$. Show that each $v$-trail in $G$ can be extended to an Euler tour of $G$ if and only if $G - v$ is acyclic.

(O. Ore)

**3.3.6** Dominating Subgraph
A subgraph $F$ of a graph $G$ is *dominating* if every edge of $G$ has at least one end in $F$. Let $G$ be a graph with at least three edges. Show that $L(G)$ is hamiltonian if and only if $G$ has a dominating eulerian subgraph.

(F. Harary and C.St.J.A. Nash-Williams)

**3.3.7** A cycle decomposition of a loopless eulerian graph $G$ induces a family of pairs of edges of $G$, namely the consecutive pairs of edges in the cycles comprising the decomposition. Each edge thus appears in two pairs, and each trivial edge cut $\partial(v)$, $v \in V$, is partitioned into pairs. An Euler tour of $G$ likewise induces a family of pairs of edges with these same two properties. A cycle decomposition and Euler tour are said to be *compatible* if, for all vertices $v$, the resulting partitions of

i) the digraph of Figure 3.7a,
ii) the four tournaments of Figure 1.25.

c) Show that the condensation of any digraph is acyclic.

d) Deduce that:

i) every digraph has a *minimal* strong component, namely one that dominates no other strong component,
ii) the condensation of any tournament is a transitive tournament.

**3.4.7** A digraph is *unilateral* if any two vertices $x$ and $y$ are connected either by a directed $(x, y)$-path or by a directed $(y, x)$-path, or both. Show that a digraph is unilateral if and only if its condensation has a directed Hamilton path.

⋆**3.4.8** Prove Theorem 3.7.

**3.4.9** De Bruijn–Good Digraph
The *de Bruijn–Good digraph* $BG_n$ has as vertex set the set of all binary sequences of length $n$, vertex $a_1a_2\ldots a_n$ being joined to vertex $b_1b_2\ldots b_n$ if and only if $a_{i+1} = b_i$ for $1 \le i \le n-1$. Show that $BG_n$ is an eulerian digraph of order $2^n$ and directed diameter $n$.

**3.4.10** De Bruijn–Good Sequence
A circular sequence $s_1s_2\ldots s_{2^n}$ of zeros and ones is called a *de Bruijn–Good sequence* of order $n$ if the $2^n$ subsequences $s_is_{i+1}\ldots s_{i+n-1}$, $1 \le i \le 2^n$ (where subscripts are taken modulo $2^n$) are distinct, and so constitute all possible binary sequences of length $n$. For example, the sequence 00011101 is a de Bruijn–Good sequence of order three. Show how to derive such a sequence of any order $n$ by considering a directed Euler tour in the de Bruijn–Good digraph $BG_{n-1}$. (N.G. de Bruijn; I.J. Good)
(An application of de Bruijn–Good sequences can be found in Chapter 10 of Bondy and Murty (1976).)

⋆**3.4.11**

a) Show that a digraph which has a closed directed walk of odd length contains a directed odd cycle.
b) Deduce that a strong digraph which contains an odd cycle contains a directed odd cycle.

## 3.5 Cycle Double Covers

In this section, we discuss a beautiful conjecture concerning cycle coverings of graphs. In order for a graph to admit a cycle covering, each of its edges must certainly lie in some cycle. On the other hand, once this requirement is fulfilled, the set of all cycles of the graph clearly constitutes a covering. Thus, by Proposition 3.2, a graph admits a cycle covering if and only if it has no cut edge. We are interested here in cycle coverings which cover no edge too many times.

Recall that a *decomposition* is a covering in which each edge is covered exactly once. According to Veblen's Theorem (2.7), the only graphs which admit such cycle coverings are the even graphs. Thus, if a graph has vertices of odd degree, some edges will necessarily be covered more than once in a cycle covering. One is led to ask whether every graph without cut edges admits a cycle covering in which no edge is covered more than twice.

All the known evidence suggests that this is indeed so. For example, each of the platonic graphs (shown in Figure 1.14) has such a cycle covering consisting of its *facial cycles*, those which bound its regions, or *faces*, as in Figure 3.8. More generally, the same is true of all polyhedral graphs, and indeed of all planar graphs without cut edges, as we show in Chapter 10.

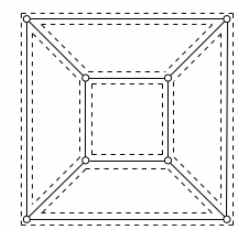

**Fig. 3.8.** A double covering of the cube by its facial cycles

In the example of Figure 3.8, observe that any five of the six facial cycles already constitute a cycle covering. Indeed, the covering shown, consisting of all six facial cycles, covers each edge exactly twice. Such a covering is called a *cycle* certain hydrocarbons by graphs and, in particular, by trees (see Exercise 4.1.3). A wide variety of proofs have since been found for Cayley's Formula (see Moon (1967)). We present here a particularly elegant one, due to Pitman (1999). It makes use of the concept of a *branching forest*, that is, a digraph each of whose components is a branching.

a) Describe a one-to-one correspondence between the set of spanning trees of $G$ that contain $e$ and the set of spanning trees of $G / e$.
b) Deduce Proposition 4.9.

**4.2.2**

a) Let $G$ be a graph with no loops or cut edges. Show that $t(G) \geq e(G)$.



b) For which such graphs does equality hold?

**4.2.3** Let $G$ be a connected graph and let $x$ be a specified vertex of $G$. A spanning $x$-tree $T$ of $G$ is called a *distance tree* of $G$ with root $x$ if $d_T(x,v) = d_G(x,v)$ for all $v \in V$.

a) Show that $G$ has a distance tree with root $x$.
b) Deduce that a connected graph of diameter $d$ has a spanning tree of diameter at most $2d$.

$\star$**4.2.4** Show that the incidence matrix of a graph is totally unimodular (defined in Exercise 1.5.7) if and only if the graph is bipartite.

**4.2.5** A *fan* is the join $P \vee K_1$ of a path $P$ and a single vertex. Determine the numbers of spanning trees in:

a) the fan $F_n$ on $n$ vertices, $n \geq 2$,
b) the wheel $W_n$ with $n$ spokes, $n \geq 3$.

**4.2.6** Let $G$ be an edge-transitive graph.

a) Show that each edge of $G$ lies in exactly $(n-1)t(G)/m$ spanning trees of $G$.
b) Deduce that $t(G \setminus e) = (m-n+1)t(G)/m$ and $t(G / e) = (n-1)t(G)/m$.
c) Deduce that $t(K_n)$ is divisible by $n$, if $n \geq 3$, and that $t(K_{n,n})$ is divisible by $n^2$.



**4.2.11** PRÜFER CODE
Let $K_n$ be the labelled complete graph with vertex set $\{1, 2, \ldots, n\}$, where $n \geq 3$. With each spanning tree $T$ of $K_n$ one can associate a unique sequence $(t_1, t_2, \ldots, t_{n-2})$, known as the *Prüfer code* of $T$, as follows. Let $s_1$ denote the first vertex (in the the ordered set $(1, 2, \ldots, n)$) which is a leaf of $T$, and let $t_1$ be the neighbour of $s_1$ in $T$. Now let $s_2$ denote the first vertex which is a leaf of $T - s_1$, and $t_2$ the neighbour of $s_2$ in $T - s_1$. Repeat this operation until $t_{n-2}$ is defined and a tree with just two vertices remains. (If $n \leq 2$, the Prüfer code of $T$ is taken to be the empty sequence.)

a) List all the spanning trees of $K_4$ and their Prüfer codes.
b) Show that every sequence $(t_1, t_2, \ldots, t_{n-2})$ of integers from the set $\{1, 2, \ldots, n\}$ is the Prüfer code of a unique spanning tree of $K_n$.
c) Deduce Cayley's Formula (see Theorem 4.8). (H. PRÜFER)

**4.2.12**

a) For a sequence $d_1, d_2, \ldots, d_n$ of $n$ positive integers whose sum is equal to $2n-2$, let $t(n; d_1, d_2, \ldots, d_n)$ denote the number of trees on $n$ vertices $v_1, v_2, \ldots, v_n$ such that $d(v_i) = d_i$, $1 \leq i \leq n$. Show that

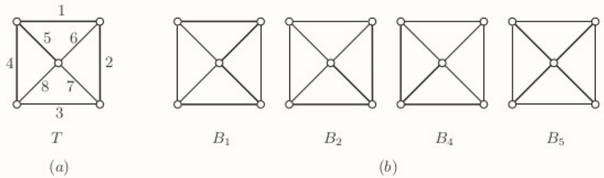


**Fig. 4.5.** (a) A spanning tree $T$ of the wheel $W_4$, and (b) the fundamental bonds with respect to $T$

The proofs of the following theorem and its corollaries are similar to those of Theorem 4.10 and its corollaries, and are left as an exercise (Exercise 4.3.5).

**Theorem 4.13** *Let $T$ be a spanning tree of a connected graph $G$, and let $S$ be a subset of $T$. Set $B := \triangle\{B_e : e \in S\}$. Then $B$ is an edge cut of $G$. Moreover $B \cap T = S$, and $B$ is the only edge cut of $G$ with this property.* □

**Corollary 4.14** *Let $T$ be a spanning tree of a connected graph $G$. Every edge cut of $G$ can be expressed uniquely as a symmetric difference of fundamental bonds with respect to $T$.* □

**Corollary 4.15** *Every spanning tree of a connected graph is contained in a unique edge cut of the graph.* □

Corollaries 4.11 and 4.14 imply that the fundamental cycles and fundamental bonds with respect to a spanning tree of a connected graph constitute bases of its cycle and bond spaces, respectively, as defined in Section 2.6 (Exercise 4.3.6). The dimension of the cycle space of a graph is referred to as its *cyclomatic number*.

In this section, we have defined and discussed the properties of fundamental cycles and bonds with respect to spanning trees in connected graphs. All the above

**4.3.2** Tree Exchange Property
Let $G$ be a connected graph, let $T_1$ and $T_2$ be (the edge sets of) two spanning trees of $G$, and let $e \in T_1 \setminus T_2$. Show that:

a) there exists $f \in T_2 \setminus T_1$ such that $(T_1 \setminus \{e\}) \cup \{f\}$ is a spanning tree of $G$,
b) there exists $f \in T_2 \setminus T_1$ such that $(T_2 \setminus \{f\}) \cup \{e\}$ is a spanning tree of $G$.

(Each of these two facts is referred to as a *Tree Exchange Property*.)

**4.3.3** Let $G$ be a connected graph and let $S$ be a set of edges of $G$. Show that the following statements are equivalent.

a) $S$ is a spanning tree of $G$.
b) $S$ contains no cycle of $G$, and is maximal with respect to this property.
c) $S$ meets every bond of $G$, and is minimal with respect to this property.

**4.3.4** Let $G$ be a connected graph and let $S$ be a set of edges of $G$. Show that the following statements are equivalent.

a) $S$ is a cotree of $G$.
b) $S$ contains no bond of $G$, and is maximal with respect to this property.
c) $S$ meets every cycle of $G$, and is minimal with respect to this property.

**4.3.5**

a) Prove Theorem 4.13.
b) Deduce Corollaries 4.14 and 4.15.

**4.3.6**

a) Let $T$ be a spanning tree of a connected graph $G$. Show that:
   i) the fundamental cycles of $G$ with respect to $T$ form a basis of its cycle space,
   ii) the fundamental bonds of $G$ with respect to $T$ form a basis of its bond space.
b) Determine the dimensions of these two spaces.

(The cycle and bond spaces were defined in Section 2.6.)

**4.3.7** Let $G$ be a connected graph, and let $\mathbf{M}$ be its incidence matrix.



a) Show that the columns of $\mathbf{M}$ corresponding to a subset $S$ of $E$ are linearly independent over $GF(2)$ if and only if $G[S]$ is acyclic.
b) Deduce that there is a one-to-one correspondence between the bases of the column space of $\mathbf{M}$ over $GF(2)$ and the spanning trees of $G$.

(The above statements are special cases of more general results, to be discussed in Section 20.2.)



Let $\mathbf{M}$ be a matrix over a field $\mathbb{F}$, let $E$ denote the set of columns of $\mathbf{M}$, and let $\mathcal{B}$ be the family of subsets of $E$ which are bases of the column space of $\mathbf{M}$. Then $(E, \mathcal{B})$ is a matroid. Matroids which arise in this manner are called *linear matroids*. Various linear matroids may be associated with graphs, one example being the matroid on the edge set of a connected graph in which the bases are the edge sets of spanning trees. (In the matroidal context, statements concerning connected graphs extend easily to all graphs, the role of spanning trees being

**5.2.10** Construct a nonseparable graph each vertex of which has degree at least four and at least two distinct neighbours, and in which splitting off any two adjacent edges results in a separable graph.

$\star$**5.2.11** Let $G$ be a nonseparable graph, and let $e$ be an edge of $G$ such that $G \setminus e$ is separable. Show that the block tree of $G \setminus e$ is a path.
(G.A. Dirac; M.D. Plummer)

**5.2.12**

a) By employing the splitting-off operation, show that every even graph has an odd number of cycle decompositions.
b) Deduce that each edge of an even graph lies in an odd number of cycles.
(S. Toida)

## 5.3 Ear Decompositions

Apart from $K_1$ and $K_2$, every nonseparable graph contains a cycle. We describe here a simple recursive procedure for generating any such graph starting with an arbitrary cycle of the graph.

Let $F$ be a subgraph of a graph $G$. An *ear* of $F$ in $G$ is a nontrivial path in $G$ whose ends lie in $F$ but whose internal vertices do not.

**Proposition 5.6** *Let $F$ be a nontrivial proper subgraph of a nonseparable graph $G$. Then $F$ has an ear in $G$.*

**Proof** If $F$ is a spanning subgraph of $G$, the set $E(G) \setminus E(F)$ is nonempty because, by hypothesis, $F$ is a proper subgraph of $G$. Any edge in $E(G) \setminus E(F)$ is then an ear of $F$ in $G$. We may suppose, therefore, that $F$ is not spanning.

Since $G$ is connected, there is an edge $xy$ of $G$ with $x \in V(F)$ and $y \in V(G) \setminus V(F)$. Because $G$ is nonseparable, $G - x$ is connected, so there is a $(y, F - x)$-path $Q$ in $G - x$. The path $P := xyQ$ is an ear of $F$. □

The proofs of the following proposition is left to the reader (Exercise 5.3.1).

**Proposition 5.7** *Let $F$ be a nonseparable proper subgraph of a graph $G$, and let $P$ be an ear of $F$. Then $F \cup P$ is nonseparable.* □

A *nested sequence* of graphs is a sequence $(G_0, G_1, \ldots, G_k)$ of graphs such that $G_i \subset G_{i+1}$, $0 \leq i < k$. An *ear decomposition* of a nonseparable graph $G$ is a nested sequence $(G_0, G_1, \ldots, G_k)$ of nonseparable subgraphs of $G$ such that:

- $G_0$ is a cycle,
- $G_{i+1} = G_i \cup P_i$, where $P_i$ is an ear of $G_i$ in $G$, $0 \leq i < k$,
- $G_k = G$.



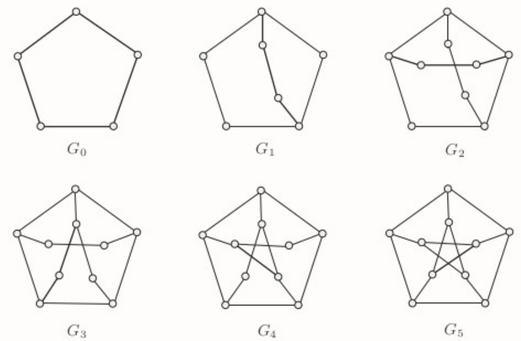


**Fig. 5.6.** An ear decomposition of the Petersen graph

An ear decomposition of the Petersen graph is shown in Figure 5.6, the initial cycle and the ear added at each stage being indicated by heavy lines.

Using the fact that every nonseparable graph other than $K_1$ and $K_2$ has a cycle, we may deduce the following theorem from Propositions 5.6 and 5.7.

**Theorem 5.8** *Every nonseparable graph other than $K_1$ and $K_2$ has an ear decomposition.* □

This recursive description of nonseparable graphs can be used to establish many of their properties by induction. We describe below an interesting application of ear decompositions to a problem of traffic flow. Further applications may be found in the exercises at the end of this section.

### Strong orientations

A road network in a city is to be converted into a one-way system, in order that traffic may flow as smoothly as possible. How can this be achieved in a satisfactory manner? This problem clearly involves finding a suitable orientation of the graph representing the road network. Consider, first, the graph shown in Figure 5.7a. No matter how this graph is oriented, the resulting digraph will not be strongly connected, so traffic will not be able to flow freely through the system, certain locations not being accessible from certain others. On the other hand, the graph of Figure 5.7b has the strong orientation shown in Figure 5.7c (one, moreover, in which each vertex is reachable from each other vertex in at most two steps).

Clearly, a necessary condition for a graph to have a strong orientation is that it be free of cut edges. Robbins (1939) showed that this condition is also sufficient. The proof makes use of the following easy proposition (Exercise 5.3.9).