Qubits:

Mathematical Notation

Outline

Bra-ket Notation

Vector Notation

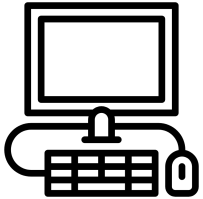
Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

Decomposing the Classical Computer

Everything is stored as a number in a variable:

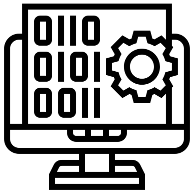
- Each *letter* of this sentence (s is 115, S is 83)
- The color of the font of this sentence
- The number of slides in this presentation
- The images included in this presentation (lots and lots of numbers)
- Sounds from an audio file



Decomposing the Classical Computer

Every number is stored in binary:

- A binary digit holds a 1 or a 0 (at any given time)
- A binary digit is called a **bit**
- 4 bits is a nibble, 8 bits is a byte



Programming languages can hide these details, providing a more intuitive programming model

Decomposing the Quantum Computer

Classical bit:

- 0 or 1

Quantum bit (qubit):

- $|0\rangle$, $|1\rangle$, or $|0\rangle$ *and* $|1\rangle$ (some probability of measuring 0 or 1)
- Phase: positive (+) or negative (-)
- ….and more (but this is all we'll cover)

Composing Computers from (Qu)bits

Classical variable:
Group of n bits stores one of 2^n possible values

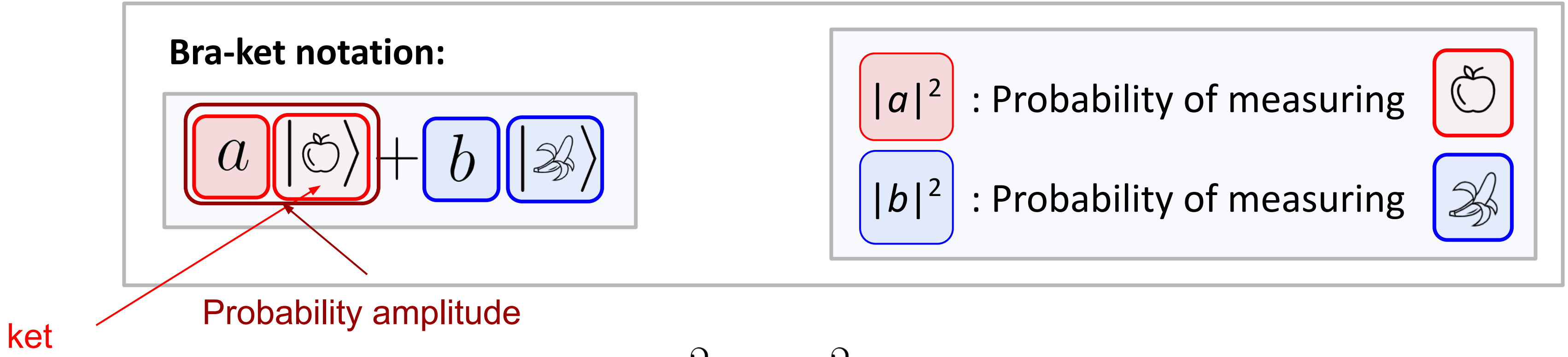
Quantum variable:
Group of n qubits stores up to 2^n possible values, with a distinct probability of measuring each individual value

This means, if I set up uneven probabilities for measuring different values,
I could use it to give a little spice / uncertainty to:

- The next slide (by slide number)
- The next letter or word that appears or the font used

Quantum State: Bra-ket Notation

Expresses **probability of measuring** each of the possible states.



Constrained by the equation: $|a|^2 + |b|^2 = 1$

Quantum State: Bra-ket Notation

Expresses **probability of measuring** 0 or 1, and indicates phase (+/-).

Bra-ket notation:

$$a|0\rangle + b|1\rangle$$

$|a|^2$: Probability of measuring 0

$|b|^2$: Probability of measuring 1

Constrained by the equation: $|a|^2 + |b|^2 = 1$

Bra-ket notation also indicates phase (+/-).

Let's relate this to the balls....

$|0\rangle$

Probability of measuring → **0 : 100%**, **1 : 0%**

Phase : Positive (+)

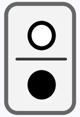
Quantum State : $1|0\rangle + 0|1\rangle$

$|1\rangle$

Probability of measuring → **0 : 0%**, **1 : 100%**

Phase : Positive (+)

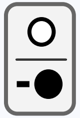
Quantum State : $0|0\rangle + 1|1\rangle$



Probability of measuring → **0 : 50%**, **1 : 50%**

Phase : Positive (+)

Quantum State : $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



Probability of measuring → **0 : 50%**, **1 : 50%**

Phase : Negative (-)

Quantum State : $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

State of Qubit

$|\psi\rangle = a|0\rangle + b|1\rangle$

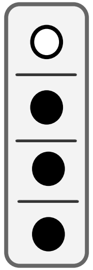
$|a|^2$: Probability measuring **0**

$|b|^2$: Probability measuring **1**

+ / - : Indicates phase

Constrained by: $|a|^2 + |b|^2 = 1$

Let’s relate this to the balls….



25% $\left(\frac{1}{4}\right)$ probability of measuring **0**

75% $\left(\frac{3}{4}\right)$ probability of measuring **1**

State of Qubit

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$|a|^2$: Probability measuring **0**

$|b|^2$: Probability measuring **1**

+ / - : Indicates phase

Constrained by: $|a|^2 + |b|^2 = 1$

$$|\psi\rangle = \frac{1}{\sqrt{4}}|0\rangle + \frac{\sqrt{3}}{\sqrt{4}}|1\rangle$$

$$|a|^2 = \left(\frac{1}{\sqrt{4}}\right)^2 = \frac{1}{4}$$

$$|b|^2 = \left(\frac{\sqrt{3}}{\sqrt{4}}\right)^2 = \frac{3}{4}$$

Probability of measuring 0 :

$\frac{1}{4}$ (or 25%)

Probability of measuring 1 :

$\frac{3}{4}$ (or 75%)

Bra-ket algebra

Like algebra, quantum notation uses conventions to improve readability.

Quantum notation simplifies in the same way as algebraic expressions.

$$z = \underbrace{15} = \underbrace{1x + 0y}$$

$$15 = 1x + \cancel{0y}$$

$$\boxed{x = 15}$$

$$|\psi\rangle = 1|0\rangle + \cancel{0|1\rangle}$$

$$\boxed{|\psi\rangle = |0\rangle}$$

Another convention is to factor out equivalent constants; I do not do this in class.

$$\cancel{\frac{1}{\sqrt{2}}}|0\rangle + \cancel{\frac{1}{\sqrt{2}}}|1\rangle = |0\rangle + |1\rangle$$

Outline

Bra-ket Notation

Vector Notation

Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

Quantum State: Vector Notation

Makes calculations of gate operations easier

Bra-ket notation:

$$a\,|0\rangle + b\,|1\rangle$$

Vector notation:

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

Vector notation:

Makes calculations of gate operations easier

$$a|0\rangle + b|1\rangle \rightarrow \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

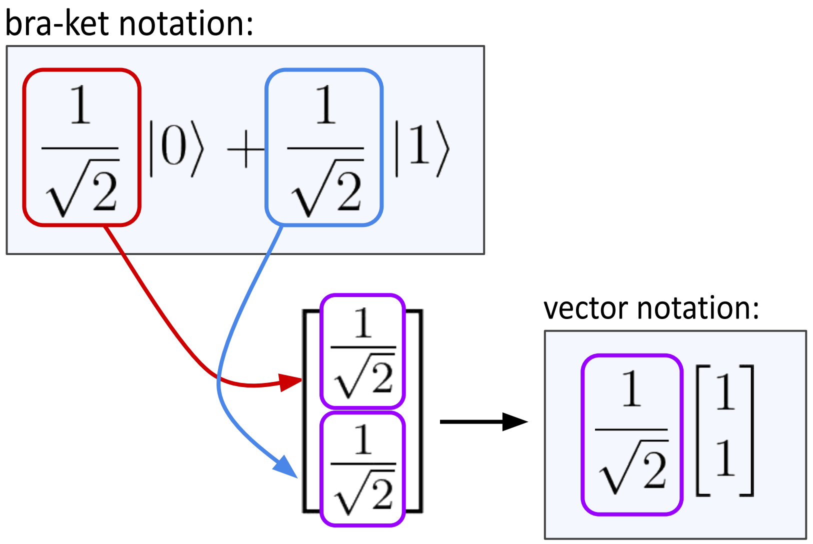
$$\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}|0\rangle \rightarrow \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

Vector notation of Quantum State:

Algebra-like simplification

Convert bra-ket notation to vector notation.

Simplify by factoring out any common constants.



Outline

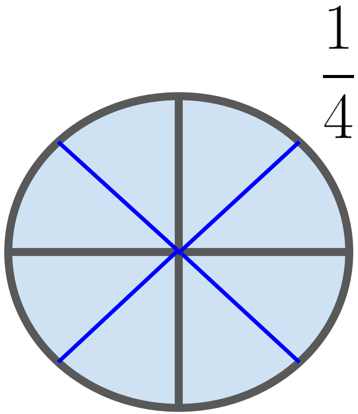
Bra-ket Notation

Vector Notation

Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

Fraction Operations



You have a pie to share with three friends.
You cut it into 4 equal pieces, each of size ¼.

Your friend exclaims, “Just because there are four of us, it doesn’t mean we need to eat the whole pie! I only want ½ that much!!”

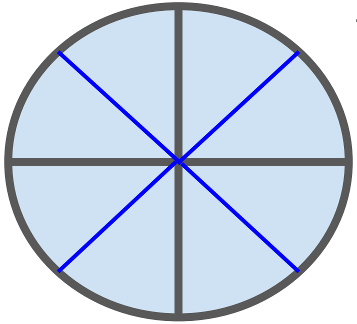
How much of the pie will you give your friend?

$$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

Representations of fractions

Fraction Operations

$\frac{1}{4}$



$$\boxed{\frac{1}{4}} \times \boxed{\frac{1}{2}} = \boxed{\frac{1}{8}}$$

$$\frac{1}{4} \times \frac{1}{2} = \frac{(1 \times 1)}{(4 \times 2)} = \frac{1}{8}$$

BUT...

$$\frac{1}{4} + \frac{1}{2} = \frac{1}{4} + \frac{2}{4} = \frac{(1 + 2)}{4} = \frac{3}{4}$$

When in doubt of the mathematics, check your work using simple examples that you can figure out with drawings!

Step 1: Figure out what the *real* operation does

Step 2: Figure out the mathematics that *always* results in the same answer

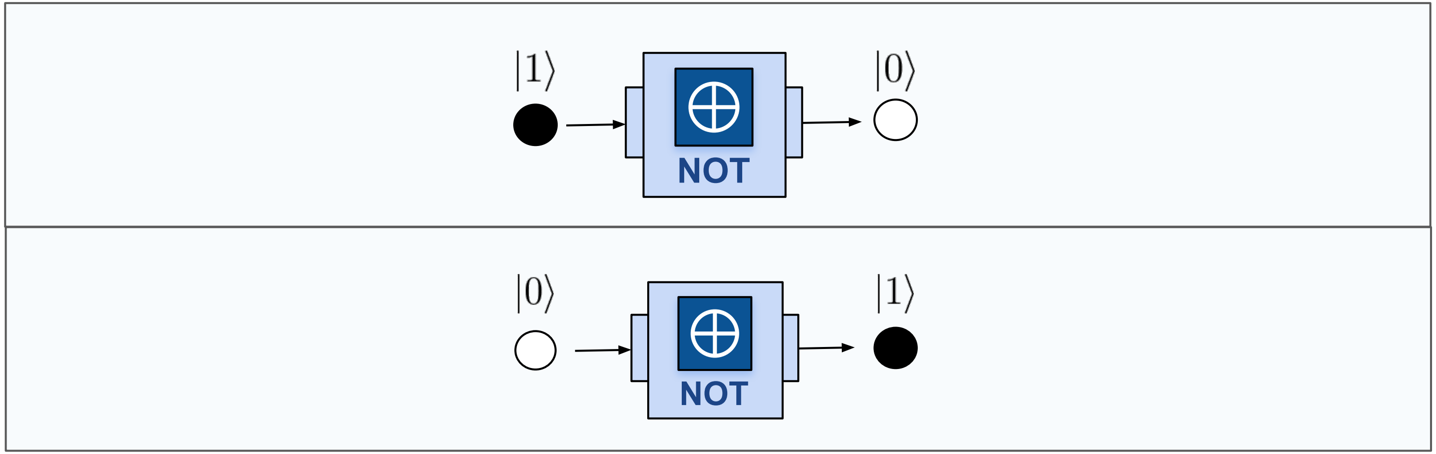
The mathematics does not always make sense on its own.

Fraction Operations: Addition & Multiplication

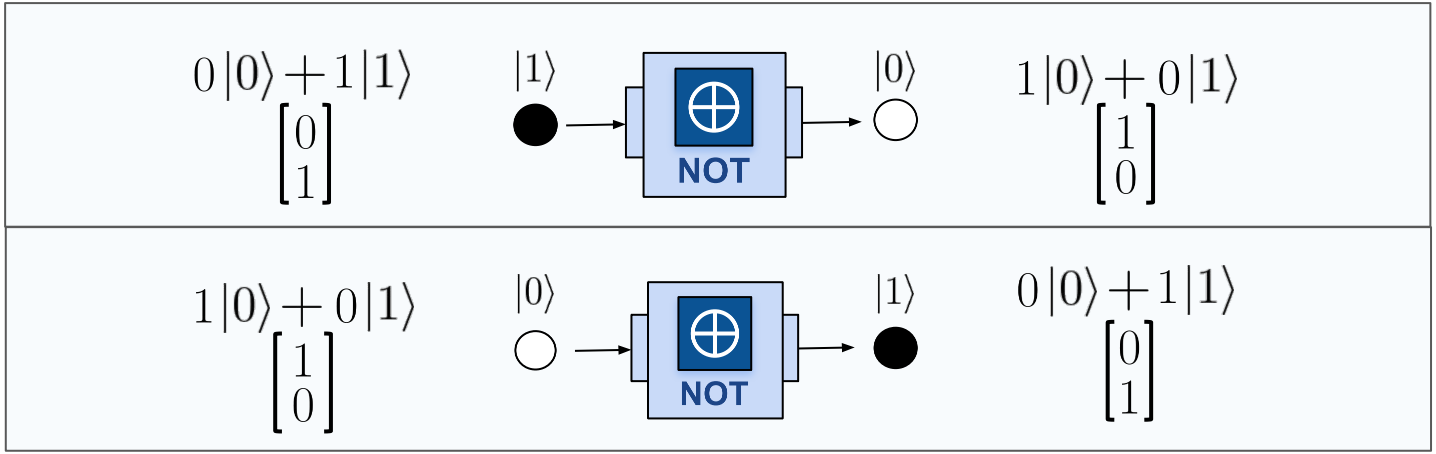
Addition with same denominators: $$\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x}$$	Example: $$\frac{1}{5}+\frac{3}{5}=\frac{1+3}{5}=\frac{4}{5}$$
Addition with different denominators: $$\frac{a}{x}+\frac{b}{y}=\frac{ay+bx}{xy}$$	Example: $$\frac{1}{3}+\frac{1}{5}=\frac{(1\times 5)}{(3\times 5)}+\frac{(1\times 3)}{(5\times 3)}=\frac{5+3}{15}=\frac{8}{15}$$
Multiplication: $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$	Example: $$\frac{1}{2}\times\frac{3}{4}=\frac{1\times 3}{2\times 4}=\frac{3}{8}$$

Have you noticed how much easier fraction multiplication is than fraction addition?

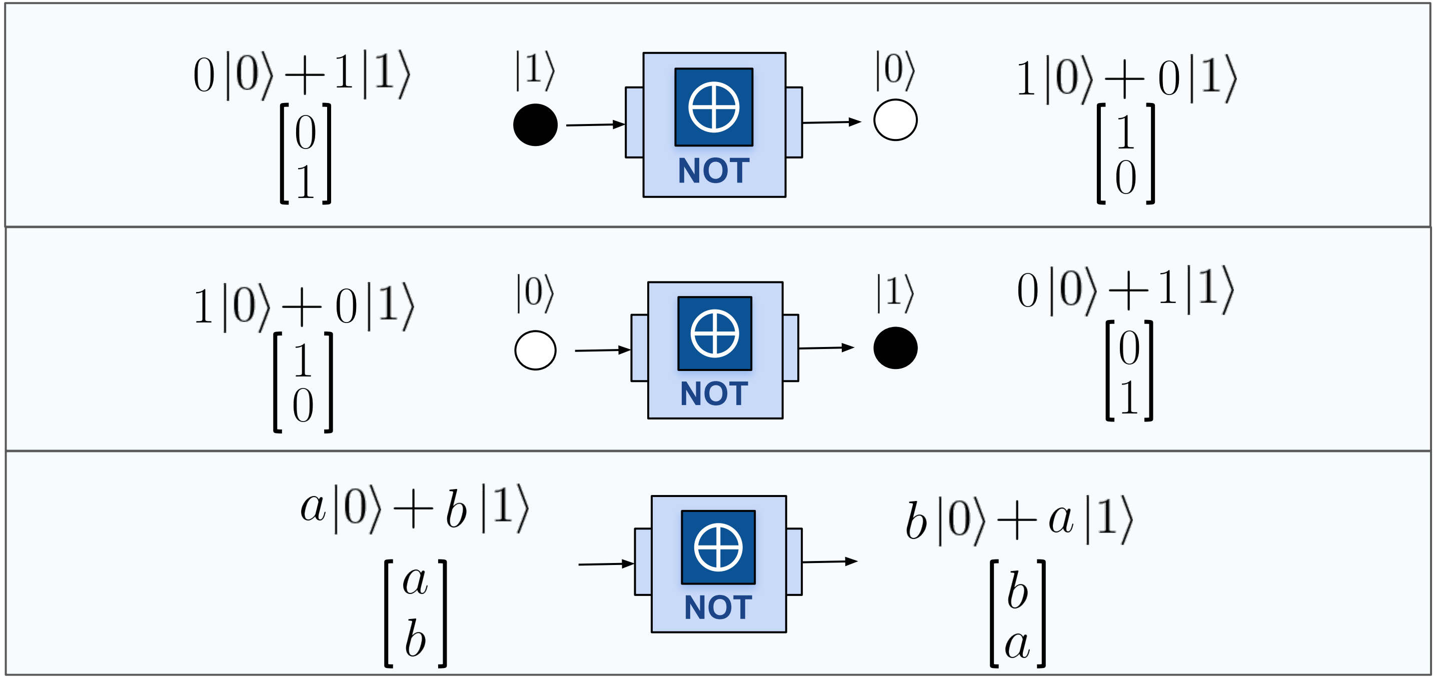
We want a mathematical calculation for….



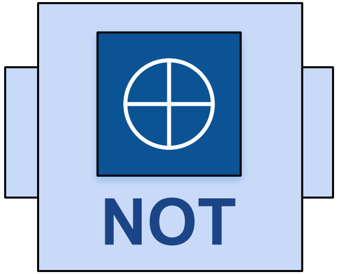
Let's revisit ways to represent qubits

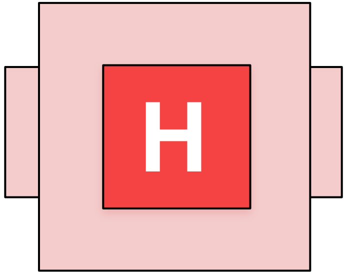


We want a mathematical calculation for ANY input



Need a similar method for all quantum gates



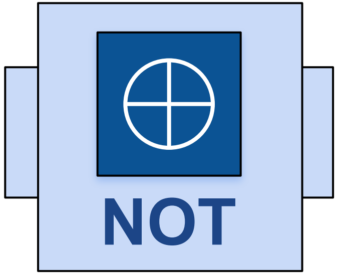


Matrix Representation

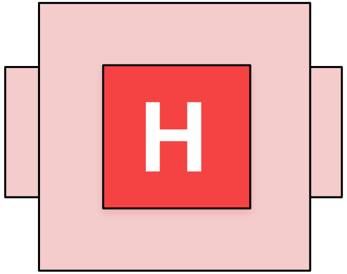
Values are laid out in a grid , similar to a spreadsheet

	Assignment 0	Assignment 1	Assignment 2	Assignment 3	Assignment 4
Student A	95	86	93	89	91
Student B	73	82	89	75	63
Student C	97	93	94	97	91
Student D	85	82	87	91	93

Quantum gates are represented as a matrix



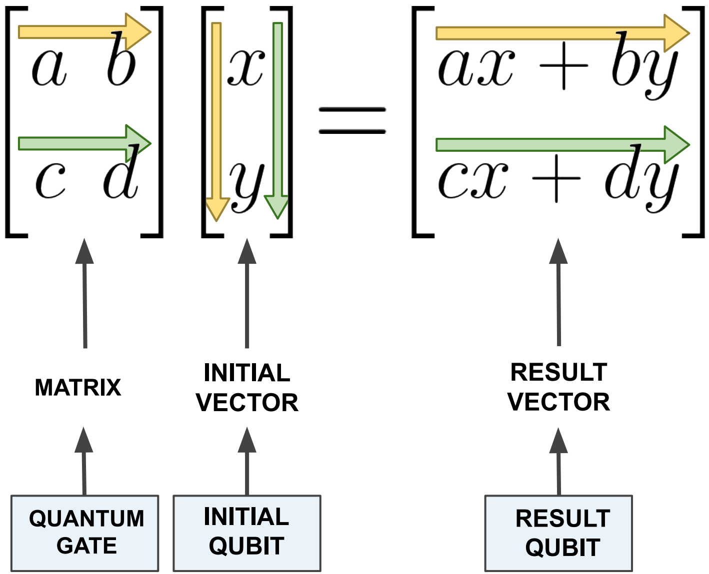
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



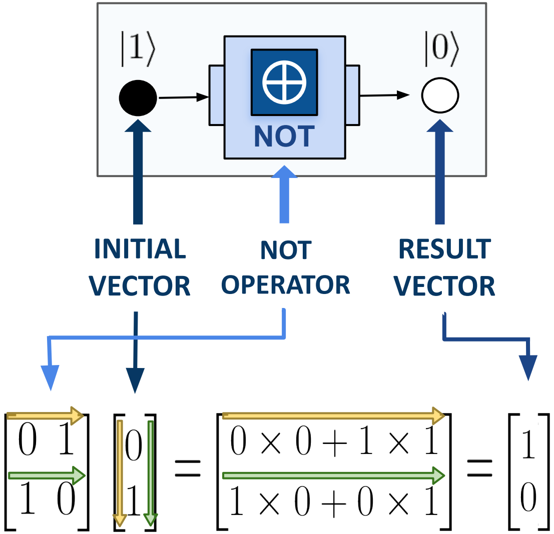
$$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

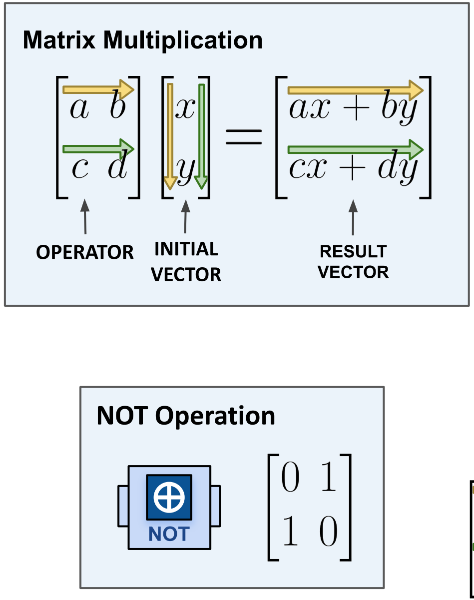
Matrix Multiplication

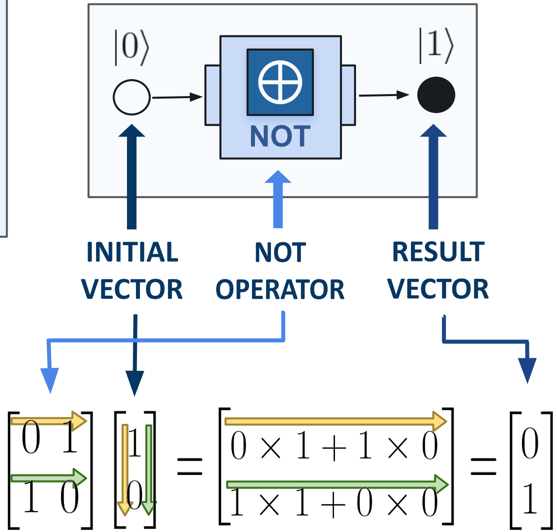
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$



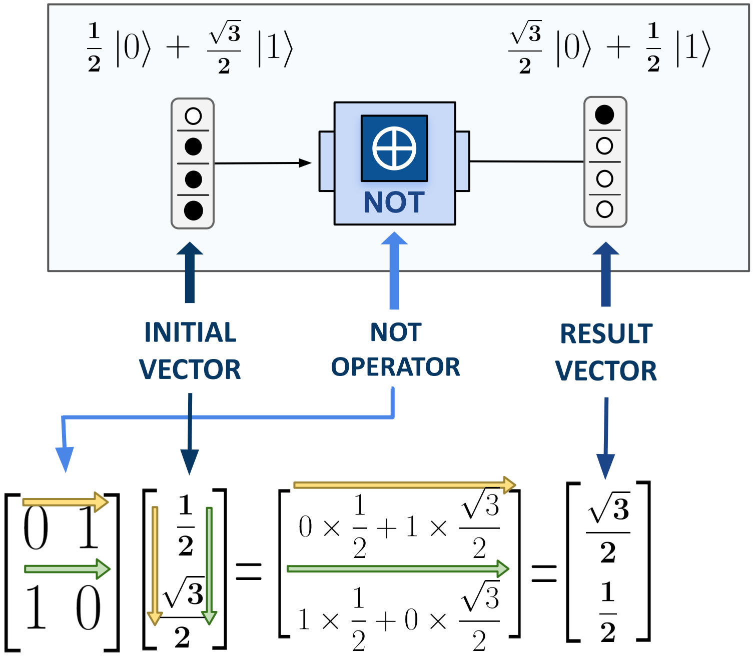
NOT Operation: Matrix Multiplication

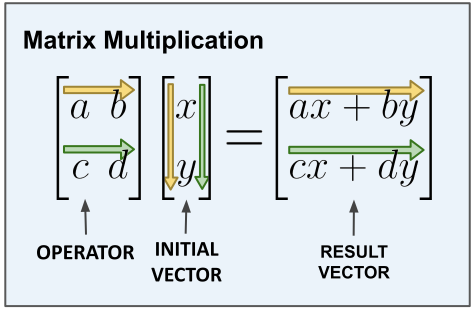


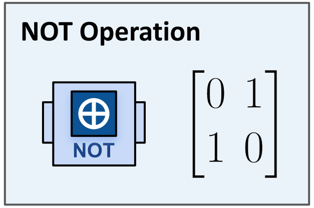




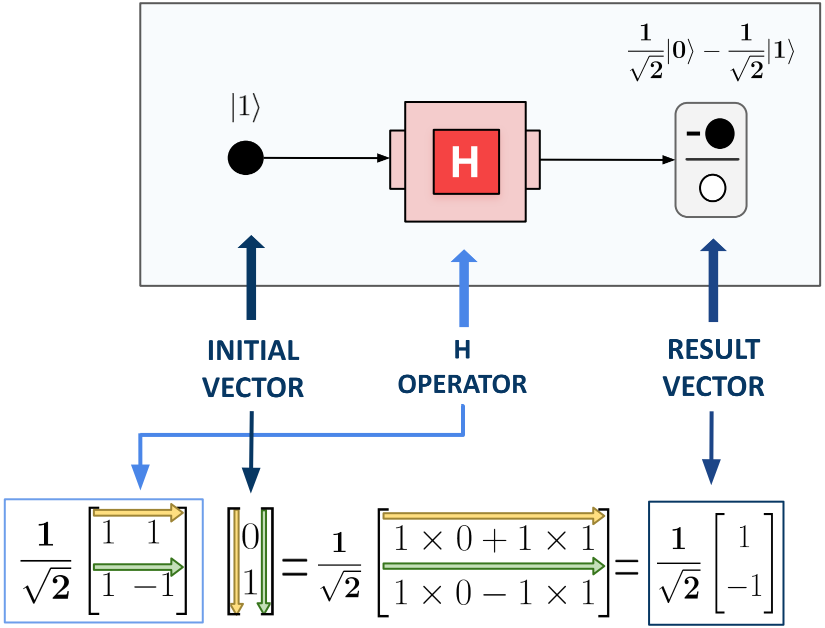
NOT Operation: Matrix Multiplication

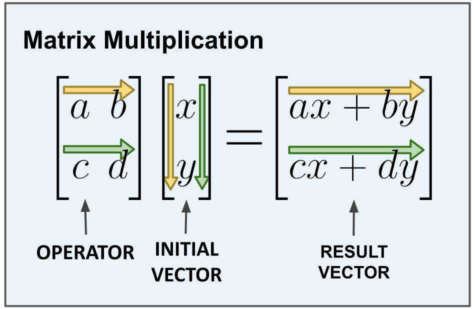


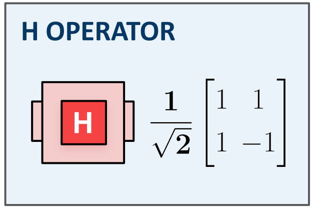




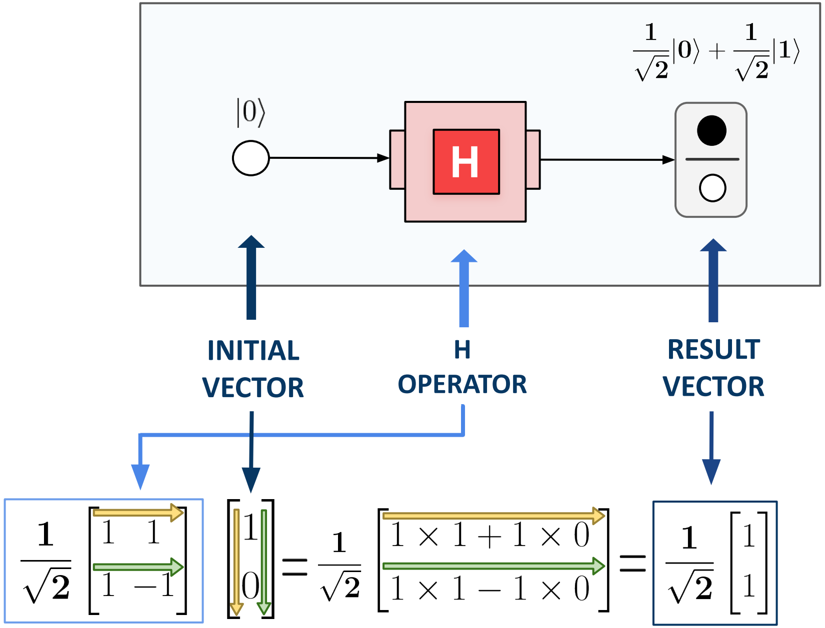
H Operation: Matrix Multiplication

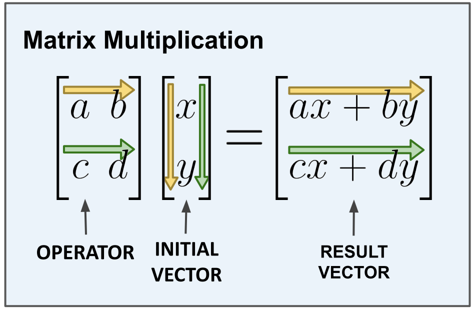


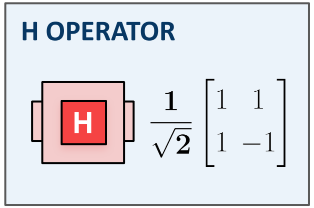




H Operation: Matrix Multiplication







Summary

- ➔ A matrix is a 2-dimensional grid of numbers in which position is important
- ➔ Each qubit operation is stored as its own unique matrix
- ➔ Matrix multiplication is used to calculate the output of a quantum operation

Outline

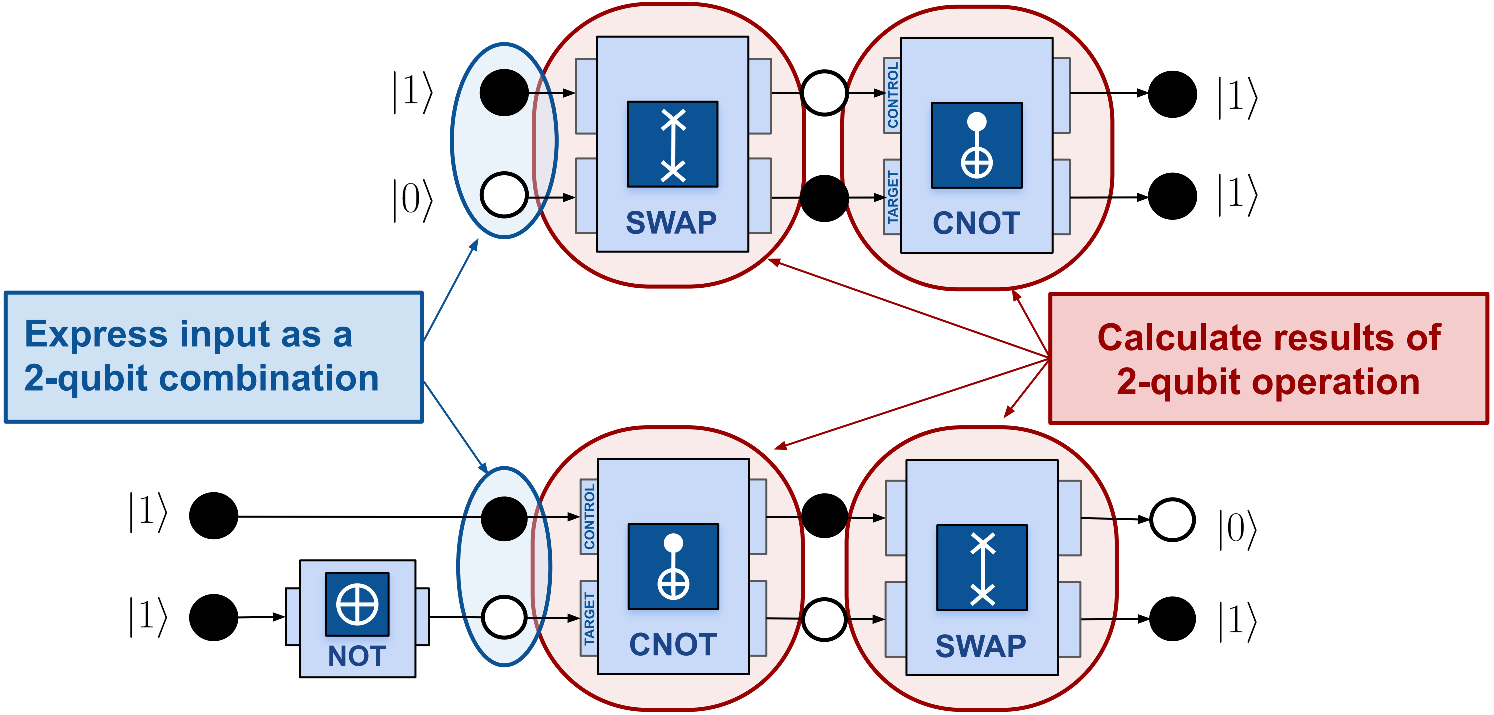
Bra-ket Notation

Vector Notation

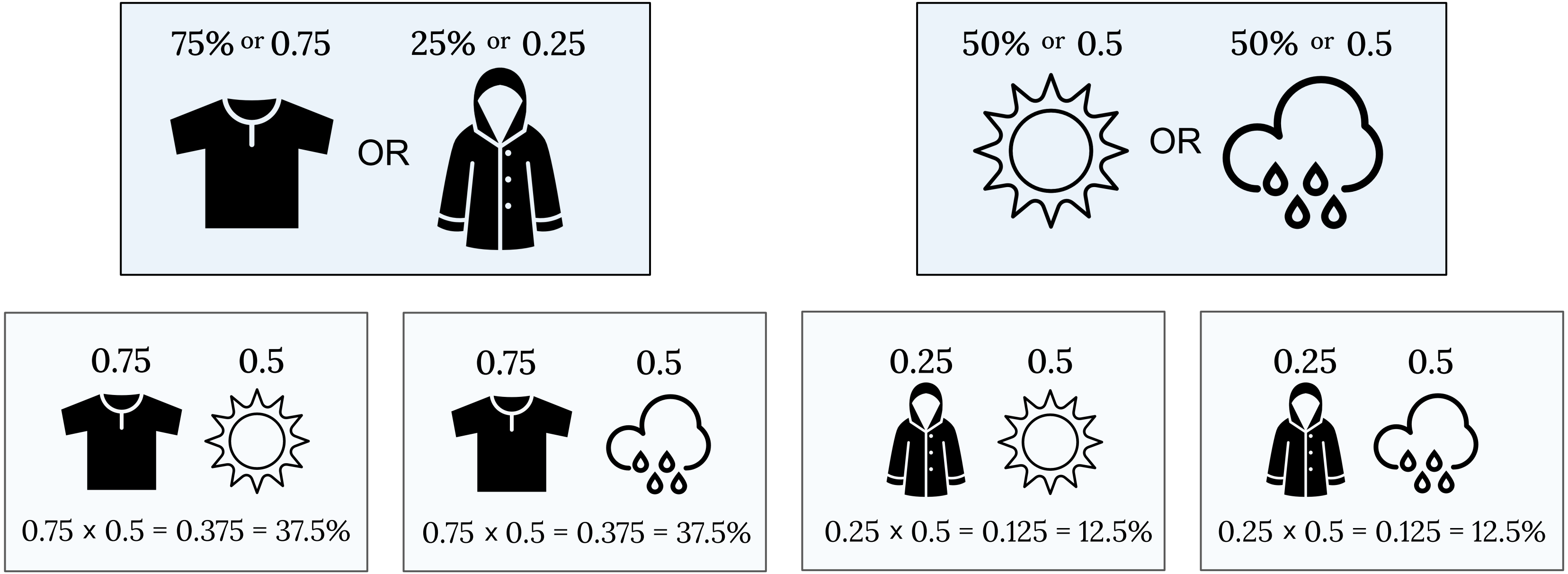
Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

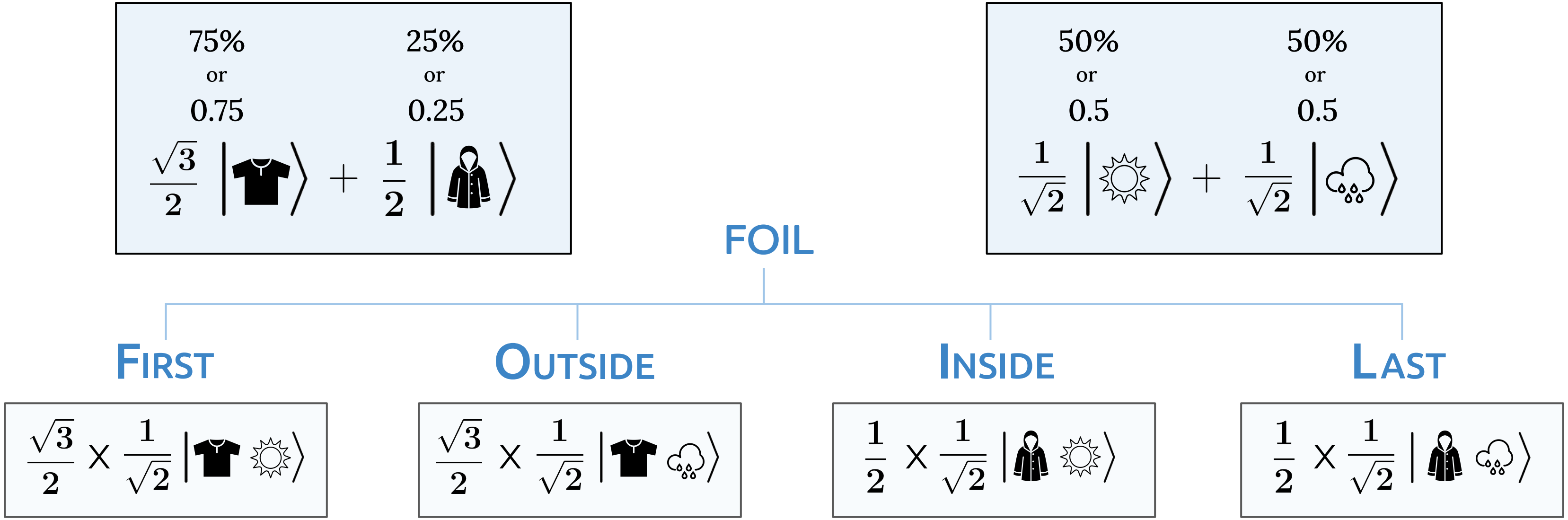
Multiple Qubit Calculations



Remember how we took the probabilities of two **independent** events and calculated the probability of different combinations of events….

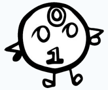


Expressing in 2-qubit bra-ket notation:

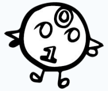


Called a **TENSOR PRODUCT** in Quantum Information Science (QIS).

Notation for Independent Qubits



TWO INDEPENDENT QUBITS



(not entangled)

1-QUBIT NOTATION

QUBIT 1: $a|0\rangle + b|1\rangle$

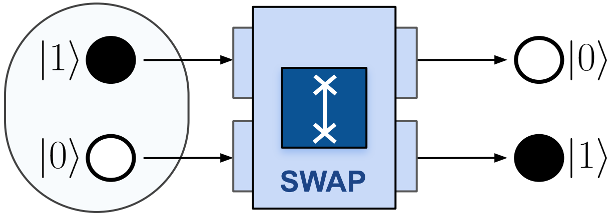
QUBIT 2: $c|0\rangle + d|1\rangle$

2-QUBIT NOTATION

QUBITS 1 & 2:

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

2-Qubit Notation: Example 1



TWO INDEPENDENT QUBITS
(not entangled)

1-QUBIT NOTATION

QUBIT 1: $a|0\rangle + b|1\rangle$

QUBIT 2: $c|0\rangle + d|1\rangle$

2-QUBIT NOTATION

QUBITS 1 & 2:

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

1-QUBIT NOTATION

$|1\rangle$ ●: $0|0\rangle + 1|1\rangle$

$|0\rangle$ ○: $1|0\rangle + 0|1\rangle$

2-QUBIT NOTATION

FOIL

FIRST OUTSIDE INSIDE LAST

$$0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle$$

VECTOR NOTATION

$$0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

2-Qubit Notation: Example 2

QUBIT 1 $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

QUBIT 2 $0|0\rangle + 1|1\rangle$

TWO INDEPENDENT QUBITS
(not entangled)

1-QUBIT NOTATION

QUBIT 1: $a|0\rangle + b|1\rangle$

QUBIT 2: $c|0\rangle + d|1\rangle$

2-QUBIT NOTATION

QUBITS 1 & 2:

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

2-QUBIT NOTATION

FOIL

FIRST **OUTSIDE** **INSIDE** **LAST**

$$\left(\frac{1}{2}\times 0\right)|00\rangle + \left(\frac{1}{2}\times 1\right)|01\rangle + \left(\frac{\sqrt{3}}{2}\times 0\right)|10\rangle + \left(\frac{\sqrt{3}}{2}\times 1\right)|11\rangle$$

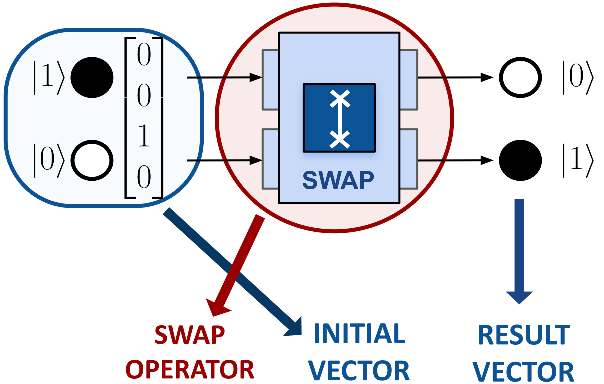
$$0|00\rangle + \frac{1}{2}|01\rangle + 0|10\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

VECTOR NOTATION

$$0\,|00\rangle + \frac{1}{2}|01\rangle + 0\,|10\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

$$\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

2-Qubit Calculation: Example 1

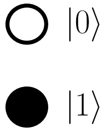


$|1\rangle$ ● , $|0\rangle$ ○ → $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ → SWAP → ○ $|0\rangle$, ● $|1\rangle$

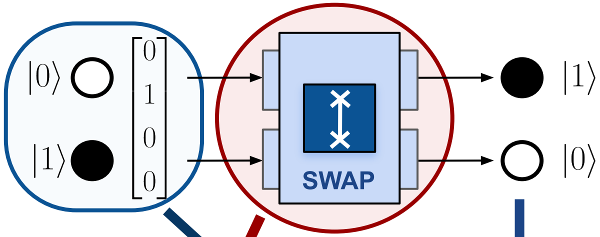
SWAP OPERATION

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=0\,|00\rangle+1\,|01\rangle+0\,|10\rangle+0\,|11\rangle=1\,|01\rangle=$$



2-Qubit Calculation: Example 2 (try yourself)

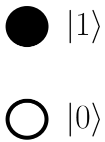


SWAP OPERATOR

INITIAL VECTOR

RESULT VECTOR

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle = 1|10\rangle =$$



SWAP OPERATION

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Intuition behind the SWAP matrix

Starting: a |00> + b|01> + c|10> + d|11>

|00> and |11> have no change when swapped

|01> -> |10>, and |01> -> |10>

Therefore, c and b should swap probabilities

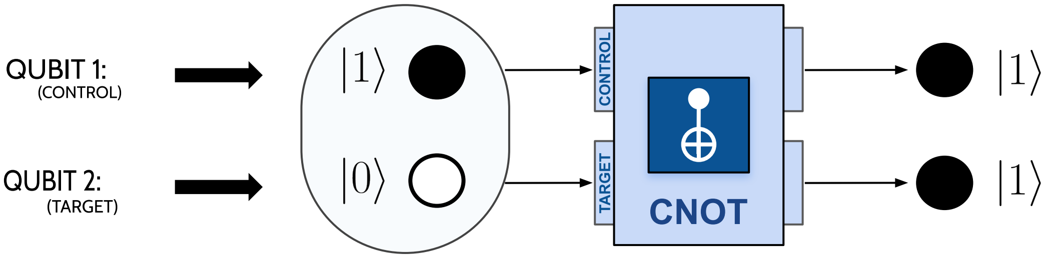
Notice the 1's on the diagonal for first and last, swap in the middle

SWAP OPERATION

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \quad \begin{bmatrix}a\\c\\b\\d\end{bmatrix}$$

2-Bit Calculation: C-NOT Gate, Order Matters!



QUBIT 1: (CONTROL)

$$0|0\rangle + 1|1\rangle$$

QUBIT 2: (TARGET)

$$1|0\rangle + 0|1\rangle$$

FOIL

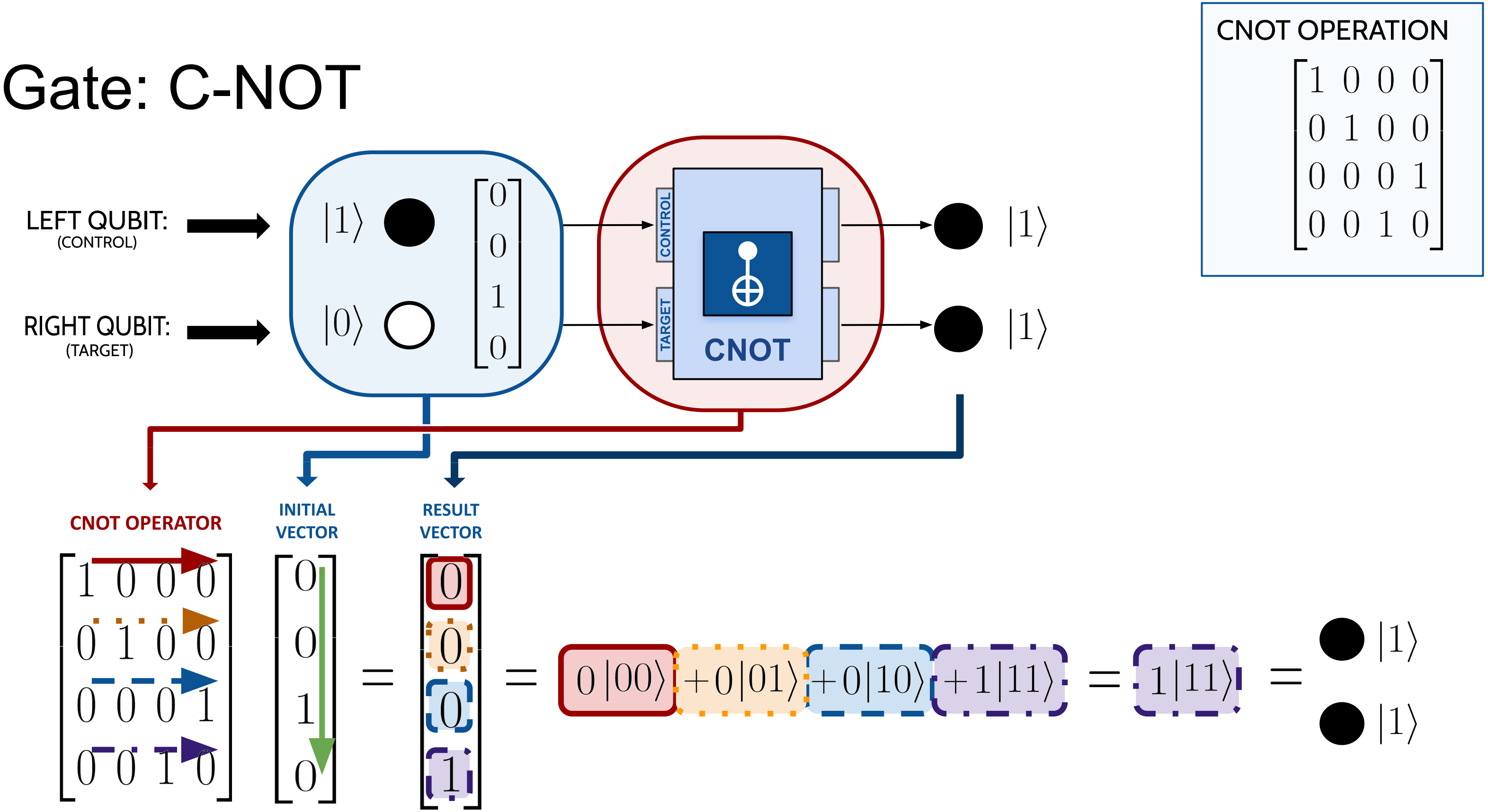
FIRST · OUTSIDE · INSIDE · LAST

$$0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle$$

INITIAL VECTOR

Control bit must always be LEFT Qubit

New Gate: C-NOT



Some intuition with the CNOT matrix

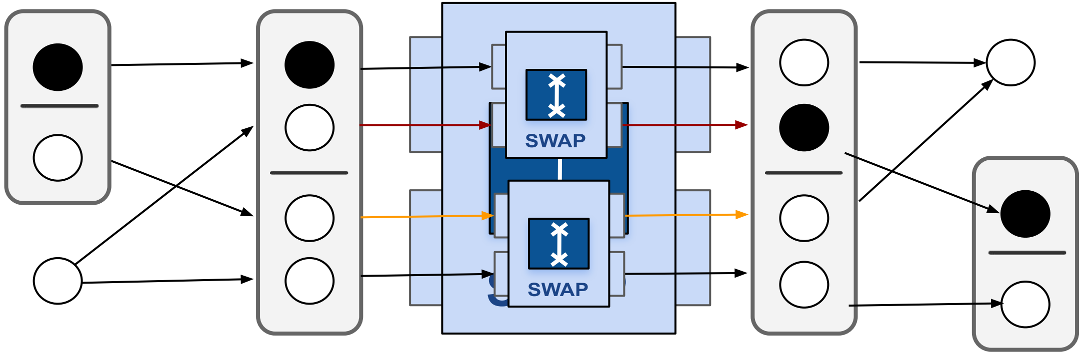
C: |0> then no change to the target
|00> -> |00>
|01> -> |01>

C: |1> then the target toggles
|10> -> |11>
|11> -> |10>

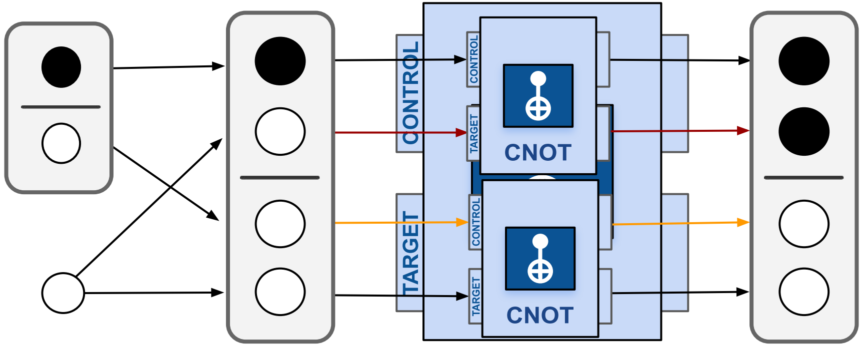
Notice where the 1's are - down the diagonal for 00 and 01, then swapping the last two.

1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

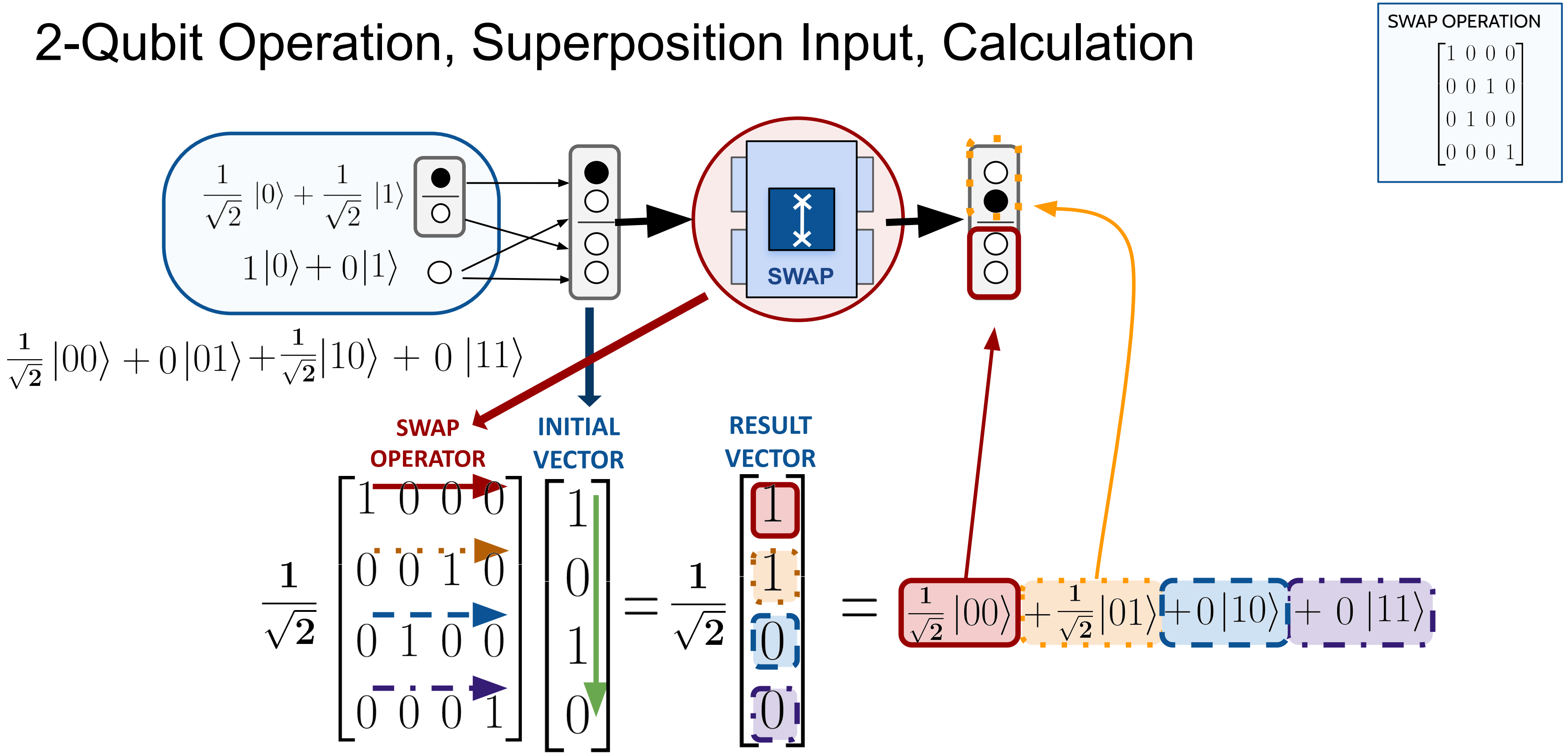
Visual Representation:
2-Qubit Operation (SWAP) & Superposition Input



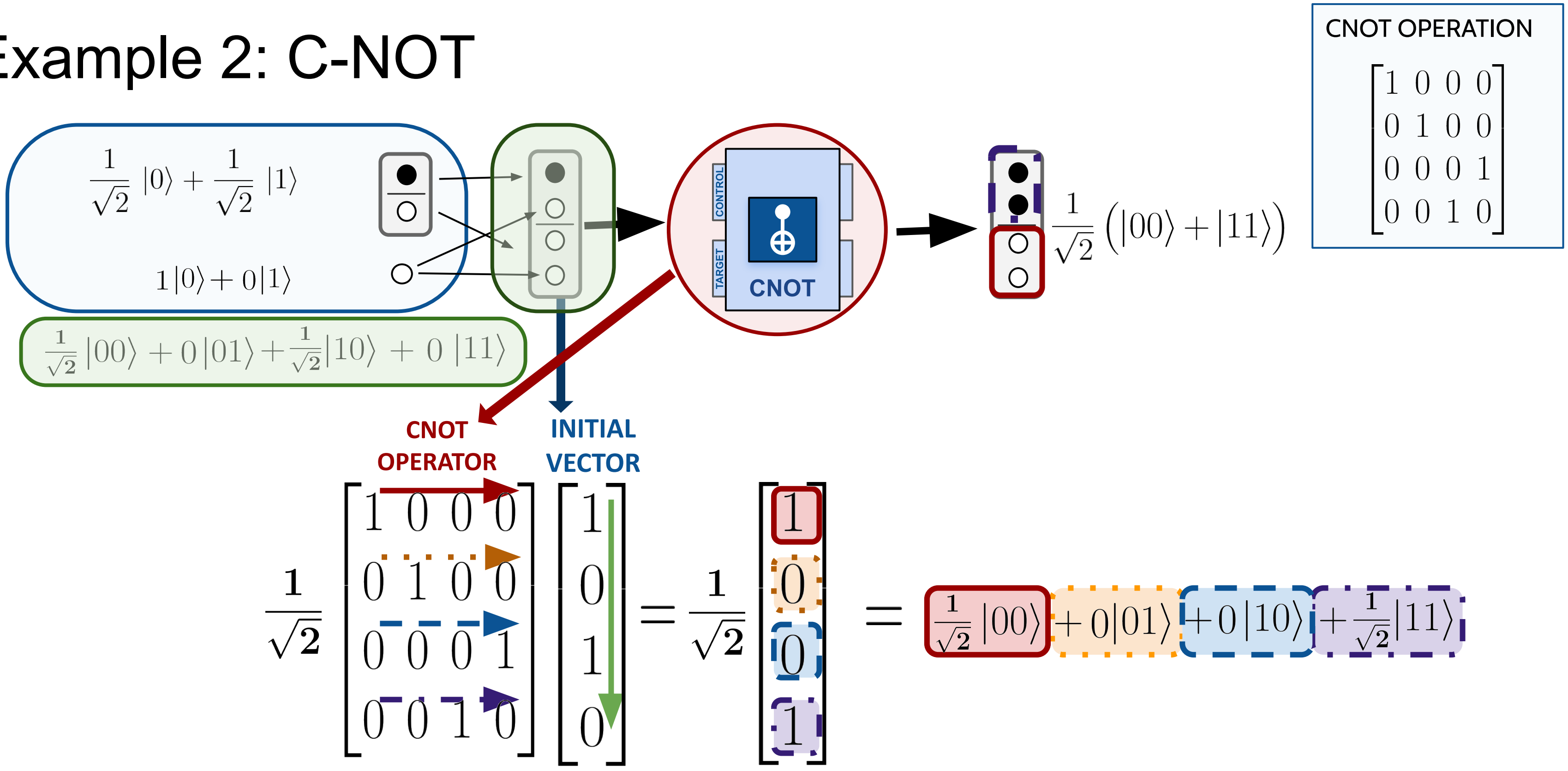
Example 2:
2-Qubit Operation (C-NOT) & Superposition Input



2-Qubit Operation, Superposition Input, Calculation



Example 2: C-NOT



Summary

To perform multiple qubit operations with inputs in superposition:

1) Put qubit state into multi-qubit notation
2) Calculate the result
    a) Visual Representation: Pass through each pair through its own gate
    b) Matrix Notation: Matrix multiplication of gate operation matrix and qubit state vector
        i) Note: This is the same as what you do when it’s not in superposition