# Qubits:

## Mathematical Notation

## Outline

**Bra-ket Notation**

Vector Notation

Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

## Decomposing the Classical Computer

Everything is stored as a number in a variable:

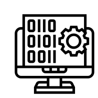
- Each *letter* of this sentence (s is 115, S is 83)
- The color of the font of this sentence
- The number of slides in this presentation
- The images included in this presentation (lots and lots of numbers)
- Sounds from an audio file

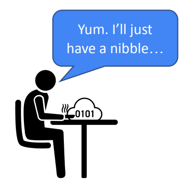


## Decomposing the Classical Computer

Every number is stored in binary:

- A binary digit holds a 1 or a 0 (at any given time)
- A binary digit is called a **bit**
- 4 bits is a nibble, 8 bits is a byte






**Programming languages can hide these details, providing a more intuitive programming model**



## Decomposing the Quantum Computer

**Classical bit**:

- 0 or 1

**Quantum bit (qubit):**

- $|0\rangle$, $|1\rangle$, or $|0\rangle$ *and* $|1\rangle$ (some probability of measuring 0 or 1)
- Phase: positive (+) or negative (-)
- ….and more (but this is all we'll cover)



## Composing Computers from (Qu)bits

**Classical variable:**
Group of n bits stores one of $2^n$ possible values

**Quantum variable:**
Group of n qubits stores up to $2^n$ possible values, with a distinct probability of measuring each individual value

This means, if I set up uneven probabilities for measuring different values,
I could use it to give a little spice / uncertainty to:

- The next slide (by slide number)
- The next letter or word that appears or the font used

# Mathematical Model

Figure out **existing mathematical symbols and operations** that will result in **efficient, accurate calculations** that **match observed results**.

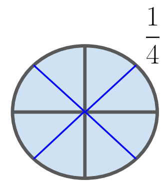
- 5 in binary: 0b00000101
- -5 in binary: 0b10000101 or 0b11111011

The former is easier for seeing the state; the latter is faster for computer computation (addition, subtraction, multiplication, division).

Quantum:

- Store probabilities
- Express three different attributes (0 vs 1, two forms of phase)
- Efficient calculations

# Fraction Operations




You have a pie to share with three friends. You cut it into 4 equal pieces, each of size ¼.

Your friend exclaims, "Just because there are four of us, it doesn't mean we need to eat the whole pie! I only want ½ that much!!"

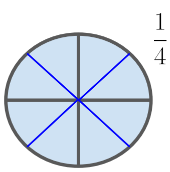
How much of the pie will you give your friend?

$$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

**Representations of fractions**



# Fraction Operations




When in doubt of the mathematics, check your work using simple examples that you can figure out with drawings!

$$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$$\frac{1}{4} \times \frac{1}{2} = \frac{(1 \times 1)}{(4 \times 2)} = \frac{1}{8}$$

BUT...

$$\frac{1}{4} + \frac{1}{2} = \frac{1}{4} + \frac{2}{4} = \frac{(1+2)}{4} = \frac{3}{4}$$

**Step 1**: Figure out what the *real* operation does

**Step 2**: Figure out the mathematics that *always* results in the same answer

The mathematics does not always make sense on its own.



# 2-bit Quantum Mathematical Model

Store probabilities:

- % <00>, % <01>, % <10> , % <11>

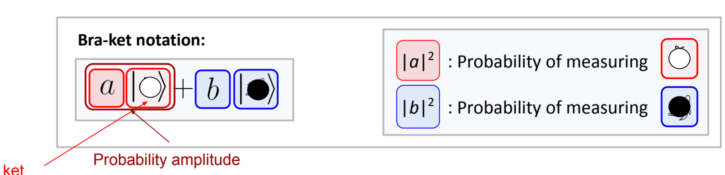
Express three independent attributes

- % <00> , % <01>, % <10> , % <11>
- % <++> , % <+->, % <-+> , % <-->
- % <aa> , % <ab>, % <ba> , % <bb>

Compact and efficient for storage and computation (next slide)

# Quantum State: Bra-ket Notation

Expresses **probability of measuring** each of the possible states.

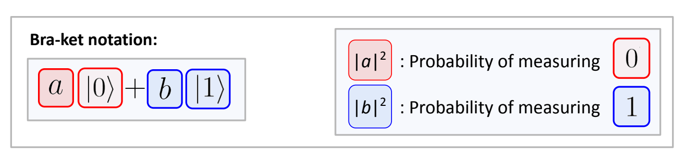



Constrained by the equation: $|a|^2 + |b|^2 = 1$



# Quantum State: Bra-ket Notation

Expresses **probability of measuring** 0 or 1, and indicates phase (+/-).

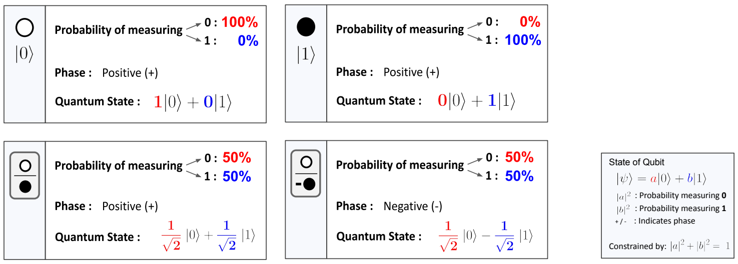



Constrained by the equation: $|a|^2 + |b|^2 = 1$

Bra-ket notation also indicates phase (+/-).

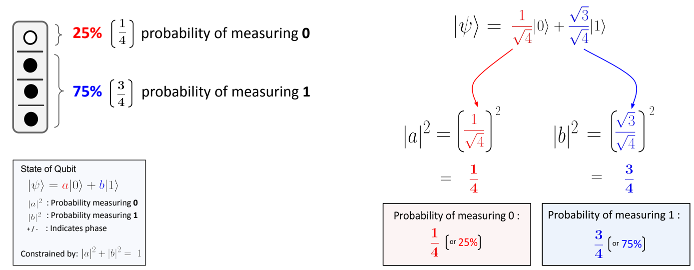
Don't confuse numbers ***inside*** brackets with numbers ***before*** brackets

## Let's relate this to the balls….






## Let's relate this to the balls….






## Bra-ket algebra

Like algebra, quantum notation uses conventions to improve readability.

Quantum notation simplifies in the same way as algebraic expressions.

$$z = 15 = 1x + 0y$$
$$15 = 1x + \cancel{0y}$$
$$\boxed{x = 15}$$

$$|\psi\rangle = 1|0\rangle + \cancel{0|1\rangle}$$
$$\boxed{|\psi\rangle = |0\rangle}$$

Another convention is to factor out equivalent constants; I do not do this in class.

$$\cancel{\frac{1}{\sqrt{2}}}|0\rangle + \cancel{\frac{1}{\sqrt{2}}}|1\rangle = |0\rangle + |1\rangle$$



## Outline

Bra-ket Notation

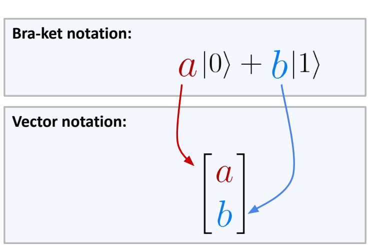
**Vector Notation**

Single-Qubit Calculations (matrix multiplication)

Multi-Qubit Calculations (single - double qubit notation)

## Quantum State: Vector Notation

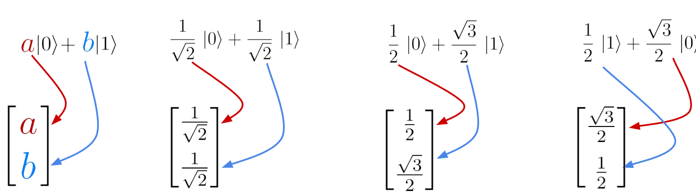
Makes calculations of gate operations easier






## Vector notation:

Makes calculations of gate operations easier

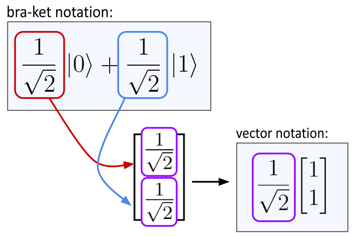


## Vector notation of Quantum State:

### Algebra-like simplification

Convert bra-ket notation to vector notation.

Simplify by factoring out any common constants.






## Outline

Bra-ket Notation

Vector Notation

**Single-Qubit Calculations (matrix multiplication)**

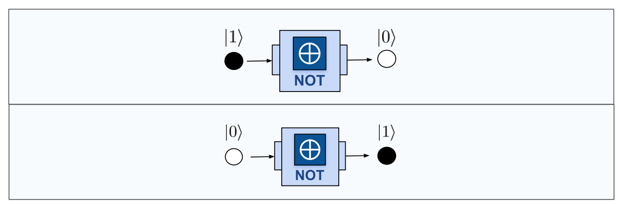
Multi-Qubit Calculations (single - double qubit notation)

## Fraction Operations: Addition & Multiplication

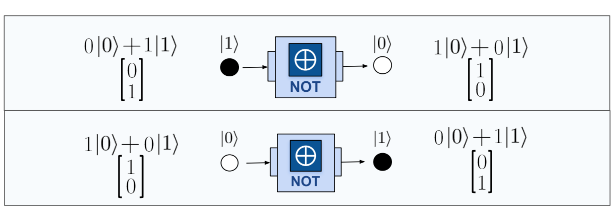
| | |
|---|---|
| Addition with same denominators: $\frac{a}{x} + \frac{b}{x} = \frac{a+b}{x}$ | Example: $\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$ |
| Addition with different denominators: $\frac{a}{x} + \frac{b}{y} = \frac{ay+bx}{xy}$ | Example: $\frac{1}{3} + \frac{1}{5} = \frac{(1\times 5)}{(3\times 5)} + \frac{(1\times 3)}{(5\times 3)} = \frac{5+3}{15} = \frac{8}{15}$ |
| Multiplication: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ | Example: $\frac{1}{2} \times \frac{3}{4} = \frac{1\times 3}{2\times 4} = \frac{3}{8}$ |

**Have you noticed how much easier fraction multiplication is than fraction addition?**

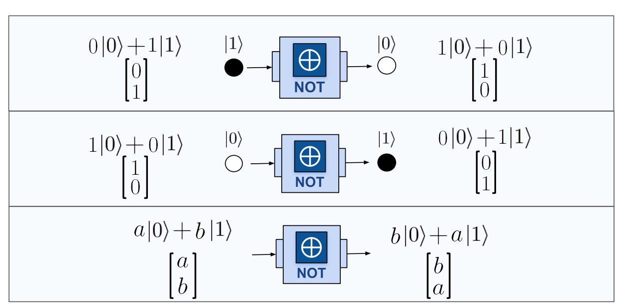
## We want a mathematical calculation for….



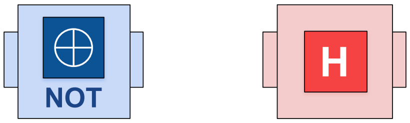

## Let's revisit ways to represent qubits




## We want a mathematical calculation for ANY input



## Need a similar method for all quantum gates

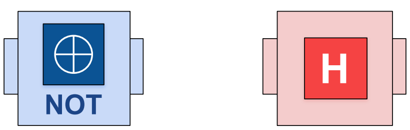



## Matrix Representation

Values are laid out in a grid, similar to a spreadsheet

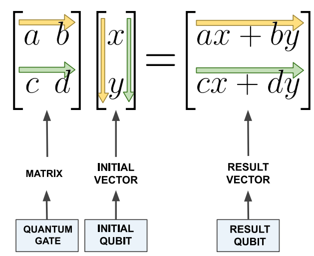
| | Assignment 0 | Assignment 1 | Assignment 2 | Assignment 3 | Assignment 4 |
|---|---|---|---|---|---|
| Student A | 95 | 86 | 93 | 89 | 91 |
| Student B | 73 | 82 | 89 | 75 | 63 |
| Student C | 97 | 93 | 94 | 97 | 91 |
| Student D | 85 | 82 | 87 | 91 | 93 |

## Quantum gates are represented as a matrix




$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

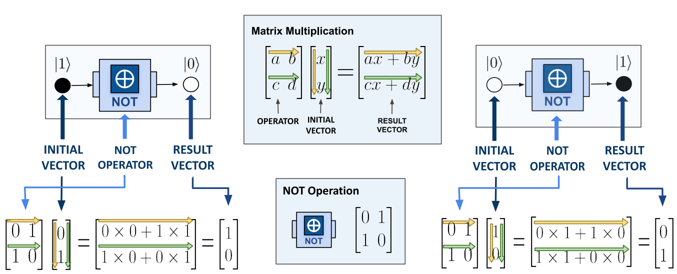
## Matrix Multiplication




$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$



## NOT Operation: Matrix Multiplication

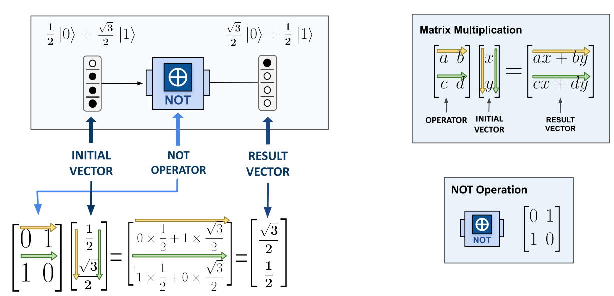



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \times 0 + 1 \times 1 \\ 1 \times 0 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 1 \times 0 \\ 1 \times 1 + 0 \times 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



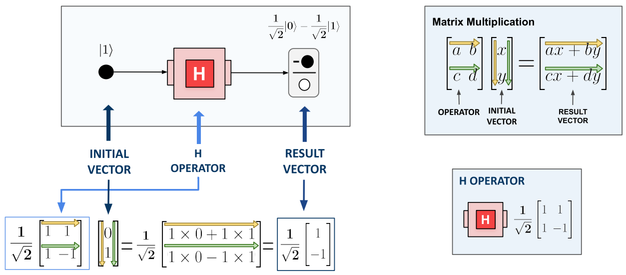
## NOT Operation: Matrix Multiplication




$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

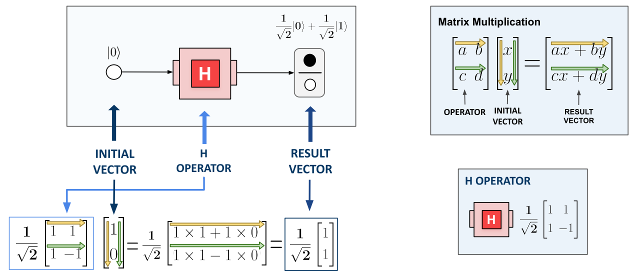
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 \times \frac{1}{2} + 1 \times \frac{\sqrt{3}}{2} \\ 1 \times \frac{1}{2} + 0 \times \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

## H Operation: Matrix Multiplication






## H Operation: Matrix Multiplication






## Summary

- ➔ A matrix is a 2-dimensional grid of numbers in which position is important
- ➔ Each qubit operation is stored as its own unique matrix
- ➔ Matrix multiplication is used to calculate the output of a quantum operation



## Outline

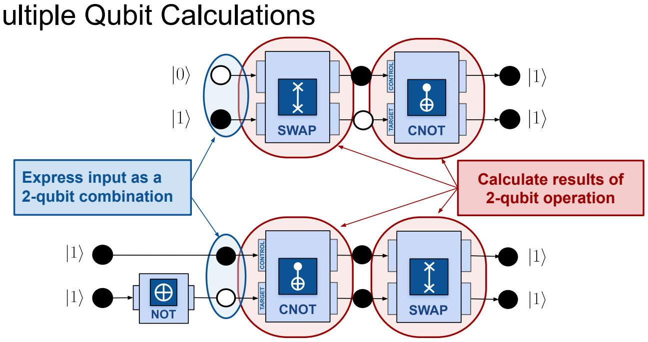
Bra-ket Notation

Vector Notation

Single-Qubit Calculations (matrix multiplication)

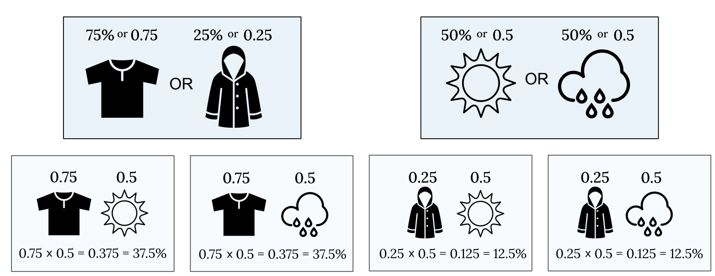
**Multi-Qubit Calculations (single - double qubit notation)**

## Multiple Qubit Calculations

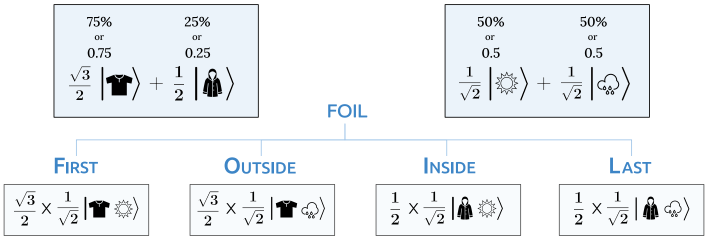





Remember how we took the probabilities of two **independent** events and calculated the probability of different combinations of events….



## Expressing in 2-qubit bra-ket notation:




Called a TENSOR PRODUCT in Quantum Information Science (QIS).

## Notation for Independent Qubits

**TWO INDEPENDENT QUBITS**
(not entangled)

**1-QUBIT NOTATION**

QUBIT 1: $a|0\rangle + b|1\rangle$

QUBIT 2: $c|0\rangle + d|1\rangle$

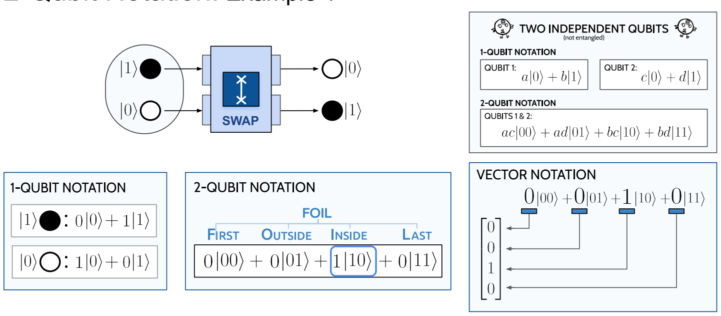
**2-QUBIT NOTATION**

QUBITS 1 & 2:

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

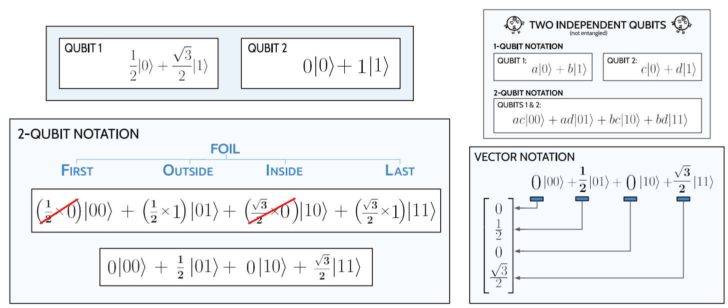


## 2-Qubit Notation: Example 1



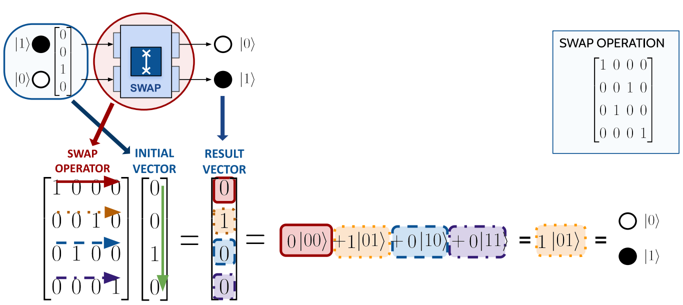



## 2-Qubit Notation: Example 2



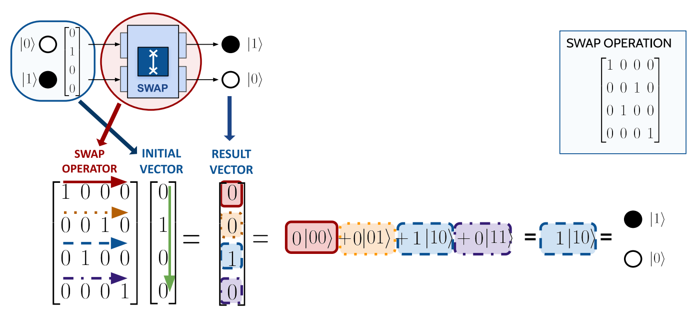



## 2-Qubit Calculation: Example 1






## 2-Qubit Calculation: Example 2 (try yourself)



## Intuition behind the SWAP matrix

Starting: a |00> + b|01> + c|10> + d|11>

|00> and |11> have no change when swapped

|01> -> |10>, and |01> -> |10>

Therefore, c and b should swap probabilities

Notice the 1's on the diagonal for first and last, swap in the middle

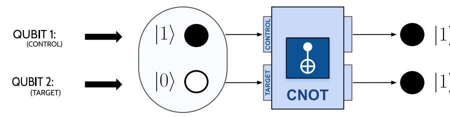
SWAP OPERATION

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\quad\begin{bmatrix}a\\c\\b\\d\end{bmatrix}$$



## 2-Bit Calculation: C-NOT Gate, Order Matters!




QUBIT 1: (CONTROL) $0|0\rangle + 1|1\rangle$

QUBIT 2: (TARGET) $1|0\rangle + 0|1\rangle$

FOIL

FIRST OUTSIDE INSIDE LAST

$$0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle$$

INITIAL VECTOR

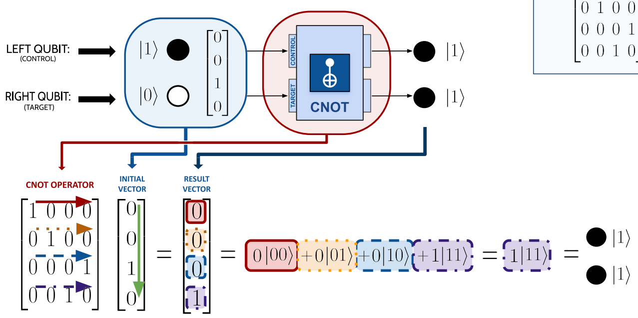
***Control bit must always be LEFT Qubit***



## New Gate: C-NOT

CNOT OPERATION

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$






## Some intuition with the CNOT matrix

C: |0> then no change to the target
|00> -> |00>
|01> -> |01>

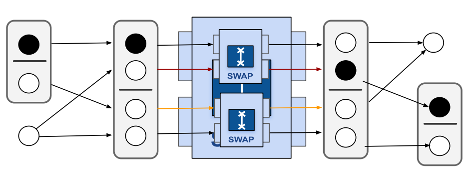
C: |1> then the target toggles
|10> -> |11>
|11> -> |10>

Notice where the 1's are - down the diagonal for 00 and 01, then swapping the last two.

| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |

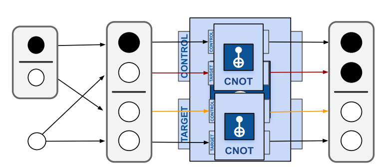


## Visual Representation:
## 2-Qubit Operation (SWAP) & Superposition Input

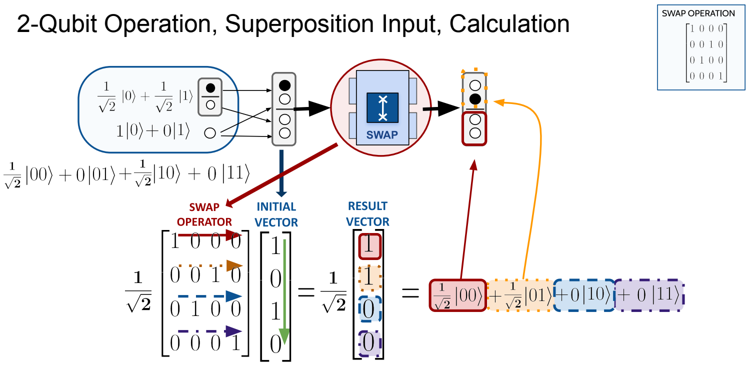





## Example 2:
## 2-Qubit Operation (C-NOT) & Superposition Input

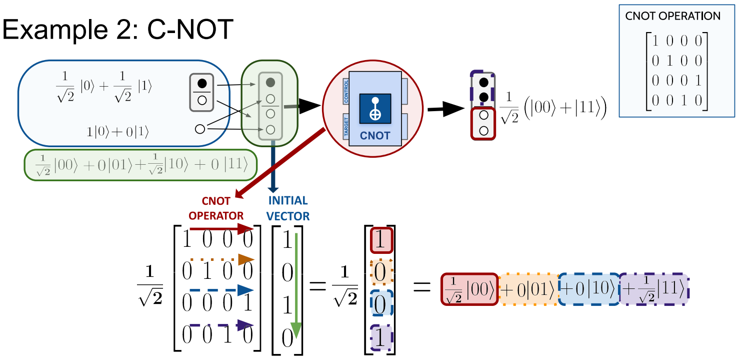


## 2-Qubit Operation, Superposition Input, Calculation






## Example 2: C-NOT






## Summary

To perform multiple qubit operations with inputs in superposition:

1) Put qubit state into multi-qubit notation
2) Calculate the result
   a) Visual Representation: Pass through each pair through its own gate
   b) Matrix Notation: Matrix multiplication of gate operation matrix and qubit state vector
      i) Note: This is the same as what you do when it's not in superposition