

Notes on polygon meshes

1 Basic definitions

Definition 1 A polygon mesh (or polymesh) is a triple (V, E, F) , where

$$\begin{array}{ll} V & \text{a set of vertices (points in space)} \\ E \subset (V \times V) & \text{a set of edges (line segments)} \\ F \subset E^* & \text{a set of faces (convex polygons)} \end{array}$$

with the following properties:

1. for any $v \in V$, there exists $(v_1, v_2) \in E$ such that $v = v_1$ or $v = v_2$.
2. for any $e \in E$, there exists a face $f \in F$ such that e is in f .
3. if two faces intersect in space, then the vertex or edge of intersection is in the mesh.

If all of the faces of a polygon mesh are triangles, then we call it a *triangle mesh* (trimesh). Polygons can be *tessellated* to form triangle meshes.

Definition 2 We classify edges in a mesh based on the number of faces they are part of:

- A boundary edge is part of exactly one face.
- An interior edge is part of two or more faces.
- A manifold edge is part of exactly two faces.
- A junction edge is part of three or more faces.

Junction edges are to be avoided; they can cause cracks when rendering the mesh.

Definition 3 A polymesh is connected if the undirected graph $G = (V_F, E_E)$, called the dual graph, is connected, where

- V_F is a set of graph vertices corresponding to the faces of the mesh and
- E_E is a set of graph edges connecting adjacent faces.

Definition 4 A polyhedron is a polymesh that is

1. *connected and*
2. *each edge is manifold.*

Definition 5 A polytope is a polyhedron that encloses a convex region R of \mathbb{R}^3 (i.e., any two points in R are connected by a line segment that is wholly contained in R).

Definition 6 A connected mesh is manifold if every edge in the mesh is either a boundary edge or a manifold edge.

For most computer graphic applications, we use manifold meshes.

Definition 7 A manifold mesh is closed if every edge is manifold and it is non-intersecting.

2 Orientation

The *orientation* of a face determines which side is the front and which side is the back. The orientation can either be Counter Clockwise (CCW, which is the OpenGL default) or Clockwise (CW).

Definition 8 Two faces, f_1 and f_2 , that share a common edge e are consistently oriented if the head of e in f_1 is the tail of e in f_2 (and vice versa).

Definition 9 A manifold mesh is orientable if the vertex orderings of its faces can be chosen so that adjacent faces have consistent orderings (i.e., all faces are either CW or CCW).

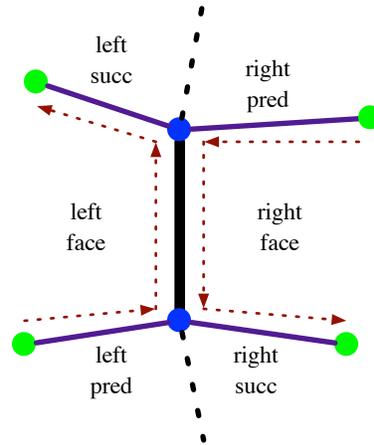
3 Data structures

The data structures used to represent meshes vary by application. For rendering purposes, one might use a wireframe representation (just vertices and edges), or a trimesh representation (vertices and triangles). If we want to do more substantial computation with the mesh, we need to be able to efficiently answer geometric queries, such as

- find the edges or vertices of a face
- find the neighboring vertices of a vertex
- find the faces of an edge
- find the edges of a vertex (c.f., silhouette edges)
- find the next edge in a path around a face

3.1 Winged-edge model

One popular representation is the *winged-edge* data structure. In this representation, the edge is the central part of the representation.



In C, we might use the following pointer-based representation:¹

```
struct edge {
    struct vertex *v0;      /* endpoints of edge */
    struct vertex *v1;
    struct face   *left;   /* face on left-hand-side of edge */
    struct face   *right;  /* face on right-hand-side of edge */
    struct edge   *lPred;  /* left-most predecessor edge */
    struct edge   *rPred;  /* right-most predecessor edge */
    struct edge   *lSucc;  /* left-most successor edge */
    struct edge   *rSucc;  /* right-most successor edge */
};

struct vert {
    struct edge *e;
    ...
};

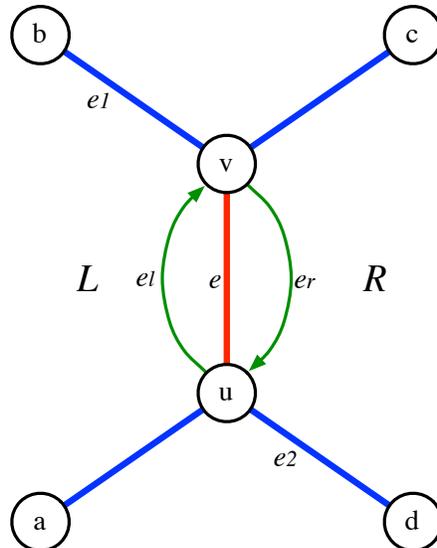
struct face {
    struct edge *e;
    ...
};
```

In a graphical application, we will store other information with vertices and faces (colors, normals, texture coordinates, ...), hence the “...” in the code. For models where the mesh is static, we can use a table-based representation that is more compact (assuming that we can use the `char` or `short` type as table indices).

¹If space is an issue, we might use 16 or 32-bit indices instead of 64-bit pointers in our representations, but pointers allow for easier dynamic manipulation of the mesh.

3.2 Directed-edge model

The *directed-edge* (or *half-edge*) model splits the representation of each edge into two oriented parts. Consider the following fragment of a mesh:



The half-edge representation uses two half-edges (e_l and e_r) to represent the edge e . For e_l , we record the source vertex (u), the face to the left (L), the next half-edge in the CCW tour of L (e_1), and the other half of the edge (e_r). Likewise, for e_r we record v , R , e_2 , and e_l . For a vertex we record an edge that it is a source of, and for a face we record an edge that the face lies to the left of. The C data structures for half-edges look like the following:

```

struct Edge {
    Vertex    *vert;    // source vertex of edge
    Face      *face;    // face to left of edge
    Edge      *next;    // next edge in tour around face
    Edge      *pair;    // half-edge going the other way
};

struct Vertex {
    Edge      *edge;    // An edge with this vertex as its source
    vec3f     v;        // The position of the vertex
    ...
};

struct Face {
    Edge      *edge;    // An edge of the face
    ...
};

```

In this representation, boundary edges will have a null pair pointer.

When restricted to triangle meshes, the directed-edge model can be made very space efficient (at a slight cost in time). Since each half-edge belongs to exactly one triangle, we can group them in triplets and use arithmetic to determine the triangle of an edge, the edges of a triangle, and the

next and previous edges of a tour:

$$\begin{aligned}\text{tri}(e) &= e \text{ div } 3 \\ \text{edges}(t) &= (3t, 3t + 1, 3t + 2) \\ \text{prev}(e) &= \begin{cases} e + 2 & \text{if } e \bmod 3 = 0 \\ e - 1 & \text{otherwise} \end{cases} \\ \text{next}(e) &= \begin{cases} e - 2 & \text{if } e \bmod 3 = 2 \\ e + 1 & \text{otherwise} \end{cases}\end{aligned}$$

The edge representation can then become very compact:

```
struct Edge {  
    struct Vertex *vert;    // source vertex of edge  
    int          pair;     // half-edge going the other way  
};
```