CMSC 28100-1 / MATH 28100-1
Introduction to Complexity Theory
Fall 2017 – Homework 6

November 4, 2017

Exercise 1. Let $\mathbf{P}$ be the complexity class of all languages decidable in polynomial time, that is, we have $\mathbf{P} = \bigcup_{k \geqslant 1} \mathrm{TIME}(n^k)$. Prove that $\mathbf{P} \neq \mathrm{TIME}(n^3)$. (Hint: use the Time Hierarchy Theorem.)

Exercise 2. The number of *head reversals* of a single-tape Turing machine M on input x is the number of times the tape head changes direction, that is, it was moving left and it moves right, or vice-versa. (Not moving does not count as reversing direction, but e.g. moving right, not moving then moving left does.)

Let $\mathrm{REVERSAL}(r(n))$ denote the class of languages decidable by Turing machines that on inputs of length n use at most $r(n)$ head reversals.

A function $R\colon \mathbb{N} \to \mathbb{N}$ is said to be *fully reversal constructible* if there exists a Turing machine M_R that always halts and takes exactly $R(n)$ head reversals on inputs of length n.

Prove the following head reversal hierarchy theorem: if $\lim_{n\to\infty} r(n)^2/R(n) = 0$ and R is fully reversal constructible, then there exists a language in $\mathrm{REVERSAL}(kR(n))$ that is not in $\mathrm{REVERSAL}(r(n))$ for some fixed $k > 0$. You can assume the following fact: if a Turing machine M on input w requires at most r head reversals, then the simulation of M on w can be done by a Universal Turing Machine using less than r^2 head reversals. (Hint: mimic the proof of the Time Hierarchy Theorem.)

Exercise 3 (Based on Exercise 9.13 of Sipser's "Introduction to Theory of Computation")**.** Consider the function $\mathrm{pad}\colon \Sigma^* \times \mathbb{N} \to \Sigma^*\#^*$ that is defined as follows. Let $\mathrm{pad}(s, \ell) = s\#^j$, where $j = \max\{0, \ell - |s|\}$ and $|s|$ is the length of s. Thus $\mathrm{pad}(s, \ell)$ simply adds enough copies of the new symbol # to the end of s so that the length of the resulting string is at least ℓ. For any language A and function $f\colon \mathbb{N} \to \mathbb{N}$, define the language $\mathrm{pad}(A, f(m))$ as

$$\mathrm{pad}(A, f(m)) = \{\mathrm{pad}(s, f(|s|)) : s \in A\}.$$

(a) Prove that if $A \in \mathrm{TIME}(n^6)$, then $\mathrm{pad}(A, n^2) \in \mathrm{TIME}(n^3)$.

(b) Let Fac be the language

$$\mathrm{Fac} = \{(x, y) : x, y \in \mathbb{N} \text{ are written in } \textit{binary} \text{ and } x \text{ has a non-trivial factor that is less or equal to } y\}.$$

A non-trivial factor of x is a positive divisor of x that is neither 1 nor x. Show that Fac can be decided in time $O(x(\log x)^2) = O(2^n n^2)$, where $n = |x|$ is the length of x. You can assume the following fact: given two integers x and y in binary, one can decide whether y is a factor of x in $O(n^2)$ steps, where n is the number of digits in x.

(c) Let UnaryFac be the language

$$\text{UnaryFac} = \{(x, y) : x, y \in \mathbb{N} \text{ are written in } \textit{unary} \text{ and } x \text{ has a non-trivial factor that is less or equal to } y\}.$$

Show that UnaryFac $\in$ TIME$(n(\log n)^2)$, where n is the length of x. (The *unary* representation of an integer x is the string 1^x.)

(d) Show that pad(Fac, 2^n) $\in$ TIME$(n(\log n)^2)$.

Exercise 4 (Undirected Hamiltonian Circuit is **NP**-complete)**.** Prove that the undirected Hamiltonian Circuit Problem UHC (i.e., the Hamiltonian Circuit Problem for undirected graphs) is **NP**-complete. You can assume that the Directed Hamiltonian Circuit Problem HC is **NP**-complete. (Hint: do a polynomial (many-to-one) reduction to HC. Remember to state what the input of your reduction is, what it produces and why this implies that UHC is **NP**-complete.)